Guarded Kleene Algebra with Tests

Coequations, Coinduction, and Completeness

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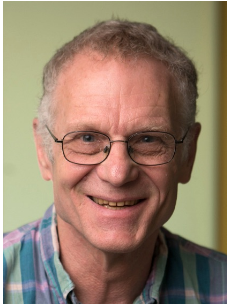
Joint work with ...

Todd Schmid
(UCL, Nijmegen)



Dexter Kozen
(Cornell)

Alexandra Silva
(Cornell, UCL)

Motivation: comparing programs

```
if not a then
    e;
else
    f;
end
```

$$\equiv$$

```
if a then
    f;
else
    e;
end
```

Motivation: comparing programs

```
if a then
   e;
   while a do
      e;
   end
end
```

$\equiv$

```
while a do
   e;
end
```

A more complicated equivalence

```
while a and b do
    e;
end
while a do
    f;
    while a and b do
        e;
    end
end
```

$\equiv$

```
while a do
    if b then
        e;
    else
        f;
    end
end
```

Research questions

- What is the minimal set of axioms?
- Are those axioms sound and complete for a model?
- Can we decide axiomatic equivalence?

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Treat `while`-programs as expressions — c.f. (Kozen and Tseng 2008).

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$$\mathbf{a}, \mathbf{b} ::= t \in T \mid \mathbf{a} + \mathbf{b} \mid \mathbf{ab} \mid \overline{\mathbf{a}} \mid 0 \mid 1$$

$$\mathbf{e}, \mathbf{f} ::= \mathbf{a} \mid p \in \Sigma \mid \mathbf{ef} \mid \mathbf{e} +_{\mathbf{a}} \mathbf{f} \mid \mathbf{e}^{(\mathbf{a})}$$

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a `or` **b**

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`false`

$$\mathbf{e}, \mathbf{f} ::= \mathbf{a} \mid p \in \Sigma \mid \mathbf{ef} \mid \mathbf{e} +_{\mathbf{a}} \mathbf{f} \mid \mathbf{e}^{(\mathbf{a})}$$

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`assert` $\mathbf{a}$

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$\mathbf{e}; \mathbf{f}$

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`if` **a** `then` **e** `else` **f**

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`while` $\mathbf{a}$ `do` $\mathbf{e}$

Some example axioms

$$\mathbf{e} +_{\mathbf{a}} \mathbf{e} \equiv \mathbf{e}$$

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$$\texttt{if } \mathbf{a} \texttt{ then } \mathbf{e} \texttt{ else assert false} = \mathbf{e} +_{\mathbf{a}} 0$$

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$$e +_a e \equiv e \qquad e +_a f \equiv f +_{\bar{a}} e \qquad \boxed{e +_a f \equiv ae +_a f} \qquad \bar{a}a \equiv 0 \qquad 0e \equiv 0$$

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\end{aligned}$$

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$\boxed{e +_a e \equiv e}$ $\quad e +_a f \equiv f +_{\bar{a}} e \quad e +_a f \equiv ae +_a f \quad \bar{a}a \equiv 0 \quad 0e \equiv 0$

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Guarded Kleene Algebra with Tests

$$\mathsf{e} +_{\mathsf{a}} \mathsf{e} \equiv \mathsf{e} \qquad \mathsf{e} +_{\mathsf{a}} \mathsf{f} \equiv \mathsf{f} +_{\overline{\mathsf{a}}} \mathsf{e} \qquad (\mathsf{e} +_{\mathsf{a}} \mathsf{f}) +_{\mathsf{b}} \mathsf{g} \equiv \mathsf{e} +_{\mathsf{ab}} (\mathsf{f} +_{\mathsf{b}} \mathsf{g}) \qquad \mathsf{e} +_{\mathsf{a}} \mathsf{f} \equiv \mathsf{ae} +_{\mathsf{a}} \mathsf{f}$$

$$\mathsf{eg} +_{\mathsf{a}} \mathsf{fg} \equiv (\mathsf{e} +_{\mathsf{a}} \mathsf{f})\mathsf{g} \qquad (\mathsf{ef})\mathsf{g} \equiv \mathsf{e}(\mathsf{fg}) \qquad 0\mathsf{e} \equiv 0 \qquad \mathsf{e}0 \equiv 0 \qquad 1\mathsf{e} \equiv \mathsf{e} \qquad \mathsf{e}1 \equiv \mathsf{e}$$

$$\mathsf{e}^{(\mathsf{a})} \equiv \mathsf{e}\mathsf{e}^{(\mathsf{a})} +_{\mathsf{a}} 1 \qquad (\mathsf{e} +_{\mathsf{a}} 1)^{(\mathsf{b})} \equiv (\mathsf{ae})^{(\mathsf{b})} \qquad \mathsf{g} \equiv \mathsf{eg} +_{\mathsf{a}} \mathsf{f} \overset{*}{\Longrightarrow} \mathsf{g} \equiv \mathsf{e}^{(\mathsf{a})}\mathsf{f}$$

Guarded Kleene Algebra with Tests

$$e +_a e \equiv e \qquad e +_a f \equiv f +_{\bar{a}} e \qquad (e +_a f) +_b g \equiv e +_{ab} (f +_b g) \qquad e +_a f \equiv ae +_a f$$

$$eg +_a fg \equiv (e +_a f)g \qquad (ef)g \equiv e(fg) \qquad 0e \equiv 0 \qquad e0 \equiv 0 \qquad 1e \equiv e \qquad e1 \equiv e$$

$$e^{(a)} \equiv ee^{(a)} +_a 1 \qquad (e +_a 1)^{(b)} \equiv (ae)^{(b)} \qquad g \equiv eg +_a f \overset{*}{\Longrightarrow} g \equiv e^{(a)}f$$

it's a bit more subtle than this. . .

Guarded Kleene Algebra with Tests

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Theorem (Smolka et al. (2020))

- $\equiv$ *is sound and complete w.r.t. a natural model.*
- $\equiv$ *is decidable in almost-linear time.*

A more complicated equivalence

```
while a and b do
    e;
end
while a do
    f;
    while a and b do
        e;
    end
end
```

$\equiv$

```
while a do
    if b then
        e;
    else
        f;
    end
end
```

$$e^{(ab)} \cdot \left(fe^{(ab)}\right)^{(a)}$$

$$(e +_b f)^{(a)}$$

Open questions

- What if we drop the axiom $\mathbf{e}0 \equiv 0$?
- How expressive is this syntax?
- Funny business with the last axiom.

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This talk:

- Answer to the first question.
- Progress towards answering the second question.
- Third problem is very hard...

The axiom $\mathbf{e}0 \equiv 0$

Intuition: "failing now is the same as failing later" ...

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...but what if the actions before failure matter?

But wait, there's more

Provable in GKAT: $\mathbf{e}^{(\mathbf{a})} \equiv \mathbf{e}^{(\mathbf{a})}\overline{\mathbf{a}}$.

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In particular,

```
while true do e end
```

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Provable in GKAT: $\mathbf{e}^{(\mathsf{a})} \equiv \mathbf{e}^{(\mathsf{a})}\overline{\mathsf{a}}$.

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$\texttt{while true do } \mathbf{e} \texttt{ end} = \mathbf{e}^{(1)}$

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In particular,

$$\begin{aligned}\texttt{while true do } \mathbf{e} \texttt{ end} &= \mathbf{e}^{(1)} \\ &\equiv \mathbf{e}^{(1)} \cdot \overline{1}\end{aligned}$$

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🤔

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See also (Mamouras 2017).

Mission statement

Question

Let $\equiv_0$ be like $\equiv$, but without relating $\mathbf{e}0$ *to* 0.

Can we recover the same results for this finer equivalence?

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Roadmap:

1. Find a model satisfying the axioms.
2. Prove soundness and completeness.
3. Decide equivalence within that model.

Guarded trees — informal description

A tree where, for each set of tests $\alpha \subseteq T$, a node either . . .

- . . . transitions to an "accept" or "reject" leaf node, or
- . . . transitions to another internal node, executing an action $p \in \Sigma$.

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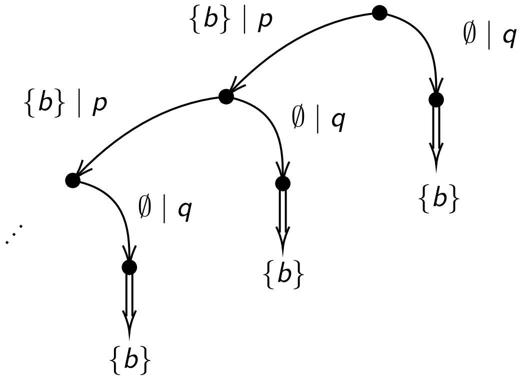
Note: guarded trees may be infinite!

Guarded trees — example

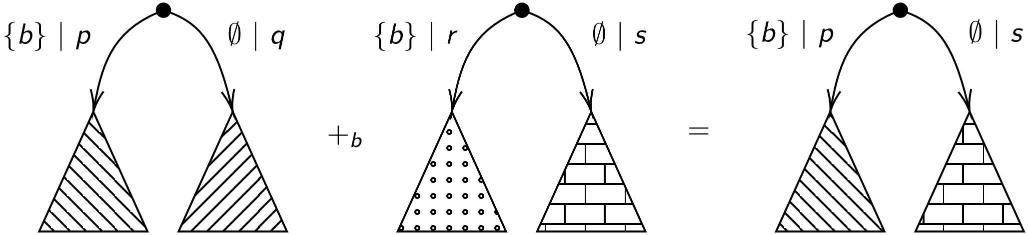


Expressions to trees — base case

$$a \;=\; \{b_0,\; b_1,\; \dots\} \;\mapsto\; \begin{tikzcd} \bullet \arrow[d, Rightarrow] \\ \{b_0,\; b_1,\; \dots\} \end{tikzcd}$$

$$p \in \Sigma \;\mapsto\; \begin{tikzcd} \bullet \arrow[d, "- \mid p"] \\ \bullet \arrow[d, Rightarrow] \\ 1 \end{tikzcd}$$

Expressions to trees — Party hat diagrams



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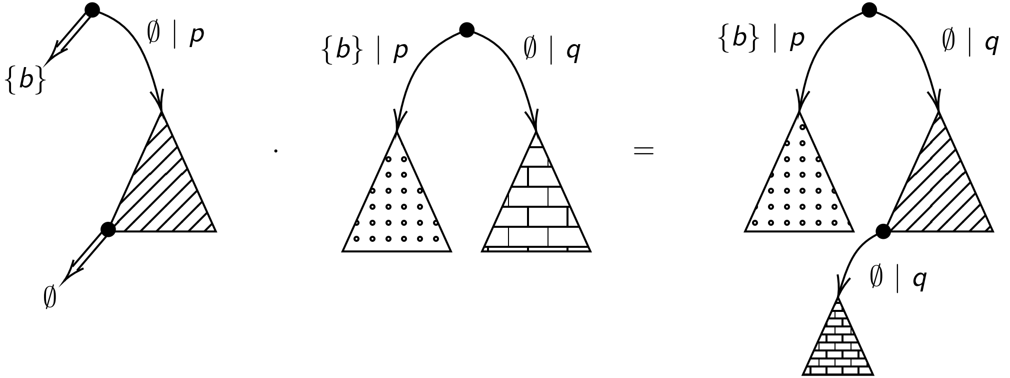

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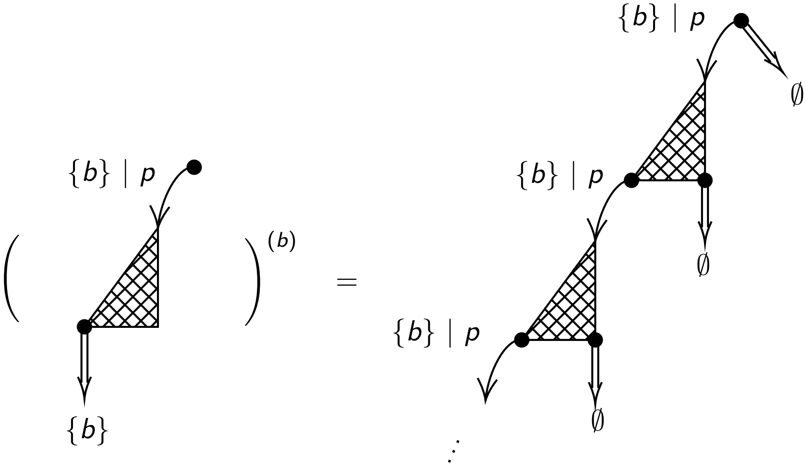

A model in terms of guarded trees

Every expression $\mathbf{e}$ has an associated guarded tree $[\![\mathbf{e}]\!]$.

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Question (Soundness & Completeness)

Is $\mathbf{e} \equiv_0 \mathbf{f}$ *equivalent to* $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

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Is $\mathbf{e} \equiv_0 \mathbf{f}$ *equivalent to* $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

Question (Decidability)

Can we decide whether $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

Establishing completeness and decidability

Theorem (Soundness & Completeness)

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Note: decision procedures are *nearly linear* — actually feasible!

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It is decidable whether $\mathbf{e} \equiv_0 \mathbf{f}$

Note: decision procedures are *nearly linear* — actually feasible!

The "old" results from (Smolka et al. 2020) can be recovered from these.

Expressiveness

Question

Let $\mathtt{t}$ *be a guarded tree with finitely many distinct subtrees.*

Is there an $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \mathtt{t}$*?*

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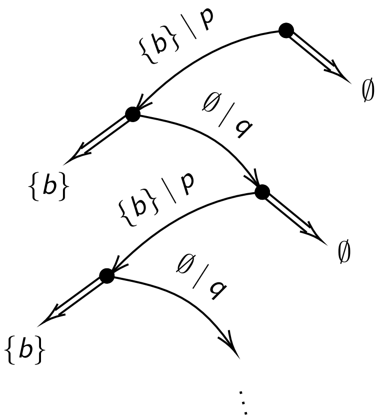
Is there an $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \texttt{t}$*?*

Not in general — for instance:



See also (Kozen and Tseng 2008).

Expressiveness

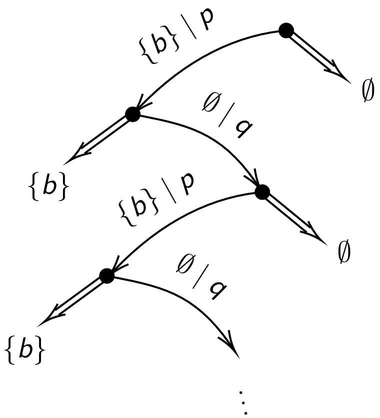
Question

Let $\texttt{t}$ *be a guarded tree with finitely many distinct subtrees.*

Is there an $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \texttt{t}$*?*

Reason: our syntax does not have `goto`.
Only *structured* programs!

Not in general — for instance:



See also (Kozen and Tseng 2008).

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Only *structured* programs!

ℓ_0 :if $\mathbf{b}$ then p; goto ℓ_1 else accept

ℓ_1 :if $\overline{\mathbf{b}}$ then q; goto ℓ_0 else accept

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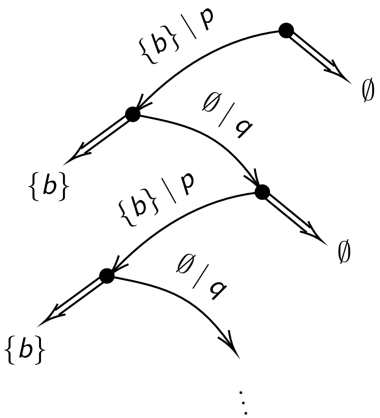

See also (Kozen and Tseng 2008).

Further work

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Is it decidable whether, given a tree $\mathtt{t}$*, there exists an* $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \mathtt{t}$*?*

Further work

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Is it decidable whether, given a tree $\texttt{t}$, *there exists an* $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \texttt{t}$?

Question

Can we identify rejection and looping without identifying early/late rejection?

What would be the appropriate axioms for such a semantics?

Thank you

https://kap.pe/slides

https://doi.org/10.4230/LIPIcs.ICALP.2021.142

Bonus — Reduction to KAT

Syntax is special case of Kleene Algebra with Tests (KAT):

$$\texttt{if } \mathbf{a} \texttt{ then } \mathbf{e} \texttt{ else } \mathbf{f} \texttt{ end} \mapsto \mathbf{a} \cdot \mathbf{e} + \overline{\mathbf{a}} \cdot \mathbf{f}$$

$$\texttt{while } \mathbf{a} \texttt{ do } \mathbf{e} \texttt{ end} \mapsto (\mathbf{a} \cdot \mathbf{e})^* \cdot \overline{\mathbf{a}}$$

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Known results:

- There is a "nice" set of axioms for KAT.
- Soundness & completeness for a straightforward model.
- Equivalence according to these axioms is decidable.

Bonus — Reduction to KAT

Equivalence in KAT is PSPACE-complete (Cohen, Kozen, and Smith 1996).

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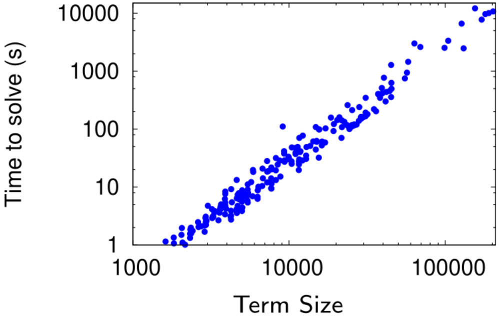
But for practical inputs, good algorithms scale well — e.g., (Foster et al. 2015):



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