Completeness and the FMP for KA, revisited

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Some context

- Laws of Kleene algebra apply to many programming language semantics.
- This means we can use KA to reason about program semantics.
- What can we (not) prove using these laws?
- When is something not true *only by the laws of KA*?

Fix a (finite) set of *letters* Σ, and write Σ^* for the set of words over Σ.

Definition (KA of languages)

The KA of *languages over* Σ is given by $(\mathcal{P}(\Sigma^*), \cup, \cdot, {}^*, \emptyset, \{\epsilon\})$, where

- $\mathcal{P}(\Sigma^*)$ is the set of sets of words (*languages*);
- $\cdot$ is pointwise concatenation, i.e., $L \cdot K = \{wx : w \in L, x \in K\}$;
- * is the Kleene star, i.e., $L^* = \{w_1 \cdots w_n : w_1, \ldots, w_n \in L\}$;
- ϵ is the empty word.

Fix a (not necessarily finite) set of *states* S.

Definition (KA of relations)

The KA of *relations over* S is given by $(\mathcal{R}(S), \cup, \circ, {}^*, \emptyset, \Delta)$, where

- $\mathcal{R}(S)$ is the set of relations on S;
- $\circ$ is relational composition.
- * is the reflexive-transitive closure.
- Δ is the identity relation.

Kleene algebra

Model theory

Let $e, f \in \text{Exp}$. We write ...

- $K, h \models e = f$ when K is a KA and $h : \Sigma \to K$ with $\widehat{h}(e) = \widehat{h}(f)$.
- $K \models e = f$ when K is a KA and $K, h \models e = f$ for all h.
- $\models e = f$ when $K \models e = f$ for every KA K.
- $\mathfrak{F} \models e = f$ when $K \models e = f$ holds in every *finite* KA K.
- $\mathfrak{R} \models e = f$ when $\mathcal{R}(S) \models e = f$ for all S.
- $\mathfrak{FR} \models e = f$ when $\mathcal{R}(S) \models e = f$ for all finite S.

Kleene algebra

Model theory

$$
\begin{array}{ccc}
\mathfrak{F} \models e = f & \overset{\text{(Palka 2005)}}{\Longleftrightarrow} & \models e = f \\
 & & \Updownarrow \text{ (Kozen 1994)} \\
\mathfrak{R} \models e = f & \overset{\text{(Pratt 1980)}}{\Longleftrightarrow} & \mathcal{P}(\Sigma^*) \models e = f
\end{array}
$$

Lemma
If $\mathfrak{FR} \models e = f$*, then* $\models e = f$*.*

Proof sketch.
We show that $\mathfrak{FR} \models e = f$ implies $\mathcal{P}(\Sigma^*) \models e = f$. If $\mathcal{P}(\Sigma^*) \not\models e = f$, then (w.l.o.g.) there exists some word x in the language of e but not in f. Choose

$$S = \{w \in \Sigma^* : |w| \leq |x|\} \qquad h : \Sigma \to \mathcal{R}(S), \mathrm{a} \mapsto \{(w, w\mathrm{a}) : w\mathrm{a} \in S\}$$

Now $\mathcal{R}(S), h \not\models e = f$. The claim then follows by Kozen's theorem. □

This talk

Palka's proof of the FMP relies on Kozen's completeness theorem.

> *... an independent proof of [the finite model property] would provide a quite different proof of the Kozen completeness theorem, based on purely logical tools. We defer this task to further research.* (Palka 2005)

We found such a proof — with many ideas inspired by Palka.

Roadmap: Given $e, f \in \mathrm{Exp}$ we do the following:

1. Turn expressions e, f into a finite automaton A
2. Turn the finite automaton A into a finite monoid M
3. Turn the finite monoid M into a finite KA K

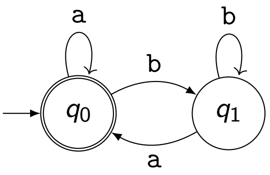
Expressions to automata

Definition

An automaton is a tuple $(Q, \to, I, F)$ where

- Q is a finite set of *states*; and
- $\to \subseteq Q \times \Sigma \times Q$ is the *transition relation*; and
- $I \subseteq Q$ is the set of *initial states*
- $F \subseteq Q$ is the set of *accepting states*

We write $q \xrightarrow{\mathtt{a}} q'$ when $(q, \mathtt{a}, q') \in \to$.



Expressions to automata

Lemma (c.f. Kleene 1956; Brzozowski 1964; Antimirov 1996)

For every e, we can construct an automaton A_e that accepts the language of e.

Automata to monoids

Let $A = (Q, \rightarrow, I, F)$ be an automaton.

Definition (Transition monoid; McNaughton and Papert 1968)

$(M_A, \circ, \Delta)$ is the monoid where $M_A = \{\xrightarrow{\mathtt{a}_1} \circ \cdots \circ \xrightarrow{\mathtt{a}_n} : \mathtt{a}_1 \cdots \mathtt{a}_n \in \Sigma^*\}$.

Monoids to Kleene algebras

Lemma (Palka 2005)

Let $(M, \cdot, 1)$ *be a monoid. Now* $(\mathcal{P}(M), \cup, \otimes, {}^{\circledast}, \emptyset, \{1\})$ *is a KA, where*

$$T \otimes U = \{t \cdot u : t \in T \wedge u \in U\} \qquad T^{\circledast} = \{t_1 \cdots t_n : t_1, \ldots, t_n \in T\}$$

Putting it all together

Given an expression e, we can now obtain a *finite* KA $K_e = \mathcal{P}(M_{A_e})$.

Lemma

Let $e, f \in \mathrm{Exp}$. *If* $K_e \models e = f$ *and* $K_f \models e = f$, *then* $\models e = f$.

Theorem (Finite model property)

If $\mathfrak{F} \models e = f$ *then* $\models e = f$.

Peeling the onion

Solving automata

Definition

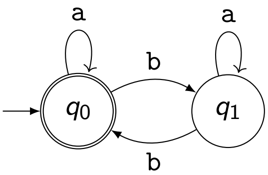
Let $(Q, \to, F)$ be an automaton. A *solution* is a function $s : Q \to \mathsf{Exp}$ such that

$$\models F(q) + \sum_{q \xrightarrow{\mathtt{a}} q'} \mathtt{a} \cdot s(q') \leq s(q) \qquad\qquad F(q) = \begin{cases} 1 & q \in F \\ 0 & q \not\in F \end{cases}$$

Example

For the automaton on the right, a solution satisfies

$$\models 1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot s(q_1) \leq s(q_0)$$
$$\models 0 + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$



E.g., $s(q_0) = (\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^*$ and $s(q_1) = \mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0)$.

Peeling the onion

Solving automata

Theorem (Kleene 1956; see also Conway 1971)

Every automaton admits a least solution (unique up to equivalence).

When A is an automaton, we write

- $\overline{A}(q)$ for the least solution to A at q
- $\lfloor A \rfloor$ for the sum of $\overline{A}(q)$ for $q \in I$

Lemma

If $e \in \mathrm{Exp}$, *then* $\models \lfloor A_e \rfloor \leq e$.

Peeling the onion

Solving monoids

Definition (Transition automaton; McNaughton and Papert 1968)

Let $R \in M_A$. We write $A[R]$ for the *transition automaton* $(M_A, \to_\circ, \{\Delta\}, \{R\})$ where

$$P \xrightarrow{\mathtt{a}}_\circ Q \iff P \circ \xrightarrow{\mathtt{a}} = Q$$

Lemma (Solving transition automata)

Let A be an automaton, let $q \in Q$ and let $R \in M_A$ with $q \, R \, q_f \in F$. We have

$$\models \lfloor A[R] \rfloor \leq \overline{A}(q)$$

Peeling the onion

Solving Kleene algebras

Let $h_e : \Sigma \to K_e$ be given by $h_e(\mathtt{a}) = \xrightarrow{\mathtt{a}}_e$.

Lemma

Let $e \in \mathrm{Exp}$ *and let* $R \in h_e(e)$*. Then* $\models \overline{A_e[R]} \leq e$*.*

Lemma

Let $e, f \in \mathrm{Exp}$*. We have that*

$$\models f \leq \sum_{R \in h_e(f)} \lfloor A_e[R] \rfloor$$

Proof sketch.

By induction on f. □

Lemma

Let $e, f \in \mathrm{Exp}$. *If* $K_e \models e = f$ *and* $K_f \models e = f$, *then* $\models e = f$.

Proof.

Since $K_e \models e = f$, we have that $h_e(e) = h_e(f)$; we can then derive

$$\models f \leq \sum_{R \in h_e(f)} \lfloor A_e[R] \rfloor = \sum_{R \in h_e(e)} \lfloor A_e[R] \rfloor \leq e$$

By a similar argument, $\models e \leq f$; the claim then follows. □

Coq formalization

- All results formalized in the Coq proof assistant.
- Trusted base:
  - Calculus of Inductive Constructions.
  - Streicher's *axiom K*.
  - Dependent functional extensionality.
- Some concepts are encoded differently; ideas remain the same.

Pomsets

Expressions in *concurrent KA* (CKA) are generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e \parallel f \mid e^* \mid e^\dagger$$

Definition (Bi-KA)

A *bi-KA* is a tuple $(K, +, \cdot, \parallel, {}^*, {}^\dagger, 0, 1)$ where

- $(K, +, \cdot, {}^*)$ and $(K, +, \parallel, {}^\dagger)$ are both KAs, and
- $\parallel$ commutes, i.e., $K \models e \parallel f = f \parallel e$.

A *weak bi-KA* is a bi-KA without the †.

Definition (Concurrent KA)

A *(weak) concurrent KA* is a (weak) bi-KA K satisfying

$$(e \parallel g) \cdot (f \parallel h) \leq (e \cdot f) \parallel (g \cdot h)$$

Pomsets

Example

The *bi-KA of pomset languages* over Σ is $(\mathcal{P}(\mathrm{Pom}(\Sigma)), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{1\})$, where

- $\mathrm{Pom}(\Sigma)$ denotes the set of pomsets over Σ;
- 1 denotes the empty pomset;
- $L \cdot L' = \{U \cdot V : U \in L, V \in L'\}$ and similarly for $\parallel$; and
- $L^* = \{1\} \cup L \cup L \cdot L \cup \cdots$ and $L^\dagger = \{1\} \cup L \cup L \parallel L \cup \cdots$.

Pomsets

Example

The *concurrent KA of pomset ideals* over Σ is $(\mathcal{I}(\Sigma), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{1\})$, where

- $\mathcal{I}(\Sigma)$ contains the pomset languages downward-closed under $\sqsubseteq$; and
- the operators are as for bi-KA, but followed by downward closure under $\sqsubseteq$.

Pomsets

Theorem (Laurence and Struth 2014)

Let e and f be (weak) concurrent KA expressions.

Now $\mathcal{P}(\mathrm{Pom}(\Sigma)) \models e = f$ if and only if $K \models e = f$ for all (weak) bi-KAs K

Theorem (Laurence and Struth 2017; K., Brunet, Silva, et al. 2018)

Let e and f be weak concurrent KA expressions.

Now $\mathcal{I}(\Sigma) \models e = f$ if and only if $K \models e = f$ for all weak CKAs K

Pomsets

Conjecture

Let e and f be concurrent KA expressions.

Now $\mathcal{I}(\Sigma) \models e = f$ if and only if $K \models e = f$ for all CKAs K

Current techniques do not work!

`<speculation>`

Pomsets

The following roadmap *might* work:

1. Translate CKA expressions to automata

   $\Rightarrow$ Pomset automata (K., Brunet, Luttik, et al. 2019)

   $\Rightarrow$ or HDAs (van Glabbeek 2004; Fahrenberg 2005; Fahrenberg et al. 2022)

2. Translate these automata to *ordered bimonoids* (Bloom and Ésik 1996)

   $\Rightarrow$ see also (Lodaya and Weil 2000; van Heerdt et al. 2021)

3. Translate bimonoids to concurrent KAs.

   $\Rightarrow$ essentially the same recipe?

Pomsets

</speculation>

Further open questions

- Can we apply these ideas to *guarded Kleene algebra with tests*?
- Does KA have a *finite relational model property*?
- Do these techniques extend to *KA with hypotheses*?
- Is there a representation theorem or duality for KA?

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