Guarded Kleene Algebra with Tests, redux

Tobias Kappé

Institute for Logic, Language and Computation, University of Amsterdam

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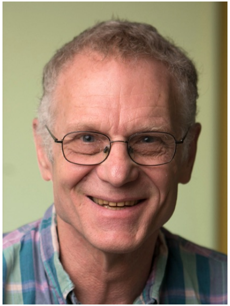
Joint work with . . .

Todd Schmid
(UCL)



Dexter Kozen
(Cornell)

Alexandra Silva
(Cornell, UCL)

Motivation: comparing programs

```
if not a then
    e;
else
    f;
end
```

$\equiv$

```
if a then
    f;
else
    e;
end
```

Motivation: comparing programs

```
if a then
   e;
   while a do
      e;
   end
end
```

$\equiv$

```
while a do
   e;
end
```

A more complicated equivalence

```
while a and b do
    e;
end
while a do
    f;
    while a and b do
        e;
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end
```

$\equiv$

```
while a do
    if b then
        e;
    else
        f;
    end
end
```

Initial questions

- What is the minimal set of axioms?
- Are those axioms complete w.r.t. some model?
- Can we decide axiomatic equivalence?

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Treat `while`-programs as expressions — c.f. (Kozen and Tseng 2008).

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$$\mathbf{a}, \mathbf{b} ::= t \in T \mid \mathbf{a} + \mathbf{b} \mid \mathbf{ab} \mid \overline{\mathbf{a}} \mid 0 \mid 1$$

$$\mathbf{e}, \mathbf{f} ::= \mathbf{a} \mid p \in \Sigma \mid \mathbf{ef} \mid \mathbf{e} +_{\mathbf{a}} \mathbf{f} \mid \mathbf{e}^{(\mathbf{a})}$$

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a `or` **b**

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`not` **a**

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`false`

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`assert` **a**

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$\mathbf{e}; \mathbf{f}$

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`if` **a** `then` **e** `else` **f**

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`while` **a** `do` **e**

Some example axioms

$$\mathbf{e} +_{\mathbf{a}} \mathbf{e} \equiv \mathbf{e}$$

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$$\texttt{if } \mathbf{a} \texttt{ then } \mathbf{e} \texttt{ else assert false} = \mathbf{e} +_{\mathbf{a}} 0$$

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$\mathbf{e} +_{\mathbf{a}} \mathbf{e} \equiv \mathbf{e}$ $\quad$ $\mathbf{e} +_{\mathbf{a}} \mathbf{f} \equiv \mathbf{f} +_{\overline{\mathbf{a}}} \mathbf{e}$ $\quad$ $\mathbf{e} +_{\mathbf{a}} \mathbf{f} \equiv \mathbf{a}\mathbf{e} +_{\mathbf{a}} \mathbf{f}$ $\quad$ $\boxed{\overline{\mathbf{a}}\mathbf{a} \equiv 0}$ $\quad$ $0\mathbf{e} \equiv 0$

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\texttt{if } \mathbf{a} \texttt{ then } \mathbf{e} \texttt{ else assert false} = \mathbf{e} +_{\mathbf{a}} 0 &\equiv \mathbf{a}\mathbf{e} +_{\mathbf{a}} 0 \\
&\equiv 0 +_{\overline{\mathbf{a}}} \mathbf{a}\mathbf{e} \\
&\equiv 0\mathbf{e} +_{\overline{\mathbf{a}}} \mathbf{a}\mathbf{e} \\
&\equiv \overline{\mathbf{a}}\mathbf{a}\mathbf{e} +_{\overline{\mathbf{a}}} \mathbf{a}\mathbf{e}
\end{aligned}$$

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$$e +_a e \equiv e \qquad e +_a f \equiv f +_{\bar{a}} e \qquad \boxed{e +_a f \equiv ae +_a f} \qquad \bar{a}a \equiv 0 \qquad 0e \equiv 0$$

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\end{aligned}$$

Guarded Kleene Algebra with Tests

$$e +_a e \equiv e \qquad e +_a f \equiv f +_{\bar{a}} e \qquad (e +_a f) +_b g \equiv e +_{ab} (f +_b g)$$

$$e +_a f \equiv ae +_a f \qquad eg +_a fg \equiv (e +_a f)g \qquad (ef)g \equiv e(fg) \qquad 0e \equiv 0$$

$$e0 \equiv 0 \qquad 1e \equiv e \qquad e1 \equiv e \qquad e^{(a)} \equiv ee^{(a)} +_a 1 \qquad (e +_a 1)^{(b)} \equiv (ae)^{(b)}$$

Guarded Kleene Algebra with Tests

Fixpoints: If $\mathbf{fe} +_{\mathbf{b}} \mathbf{g} \equiv \mathbf{e}$ and $\mathbf{e}$ is productive, then $\mathbf{f}^{(\mathbf{b})}\mathbf{g} \equiv \mathbf{e}$.

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Unique solutions: affine systems of equations, i.e., of the form

$$\begin{array}{ccccccccc}
\mathbf{e}_{1,1} \cdot x_1 & +_{\mathbf{a}_{1,1}} & \mathbf{e}_{1,2} \cdot x_2 & +_{\mathbf{a}_{1,2}} & \cdots & +_{\mathbf{a}_{1,n}} & \mathbf{b}_1 & \equiv x_1 \\
\vdots & & & & \ddots & & & \vdots \\
\mathbf{e}_{n,1} \cdot x_1 & +_{\mathbf{a}_{n,1}} & \mathbf{e}_{n,2} \cdot x_2 & +_{\mathbf{a}_{n,2}} & \cdots & +_{\mathbf{a}_{n,n}} & \mathbf{b}_n & \equiv x_n
\end{array}$$

have at most one solution (up to $\equiv$) — provided the $\mathbf{e}_{i,j}$ are *productive*.

Guarded Kleene Algebra with Tests

Theorem (Smolka et al. (2020))

- $\equiv$ *is sound and complete w.r.t. a natural model.*
- $\equiv$ *is decidable in almost-linear time.*

A more complicated equivalence

```
while a and b do
    e;
end
while a do
    f;
    while a and b do
        e;
    end
end
```

$\equiv$

```
while a do
    if b then
        e;
    else
        f;
    end
end
```

$$e^{(ab)} \cdot (fe^{(ab)})^{(a)}$$

$$(e +_b f)^{(a)}$$

Followup questions

- What if we drop the axiom $\mathbf{e}0 \equiv 0$?
- How expressive is this syntax?
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This talk:

- Answer to the first question.
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- Can we simplify the last axiom?

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Third question remains open!

The axiom $\mathbf{e}0 \equiv 0$

Intuition: “failing now is the same as failing later” . . .

The axiom $\mathbf{e}0 \equiv 0$

Intuition: "failing now is the same as failing later" ...

...but what if the actions before failure matter?

But wait, there's more

Provable in GKAT: $\mathbf{e}^{(\mathbf{a})} \equiv \mathbf{e}^{(\mathbf{a})}\overline{\mathbf{a}}$.

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In particular,

```
while true do e end
```

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In particular,

$\texttt{while true do } \mathbf{e} \texttt{ end} = \mathbf{e}^{(1)}$

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In particular,

$$\begin{aligned} \texttt{while true do } \mathbf{e} \texttt{ end} &= \mathbf{e}^{(1)} \\ &\equiv \mathbf{e}^{(1)} \cdot \overline{1} \\ &\equiv \mathbf{e}^{(1)} \cdot 0 \end{aligned}$$

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In particular,

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🤔

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See also (Mamouras 2017).

Mission statement

Question

Let $\equiv_0$ be like $\equiv$, but without relating **e**0 *to* 0.

Can we recover the same results for this finer equivalence?

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Roadmap:

1. Find a model satisfying the axioms.
2. Prove soundness and completeness.
3. Decide equivalence within that model.

Guarded trees — informal description

A tree where, for each set of tests $\alpha \subseteq T$, a node either ...

- ...transitions to an "accept" or "reject" leaf node, or
- ...transitions to another internal node, executing an action $p \in \Sigma$.

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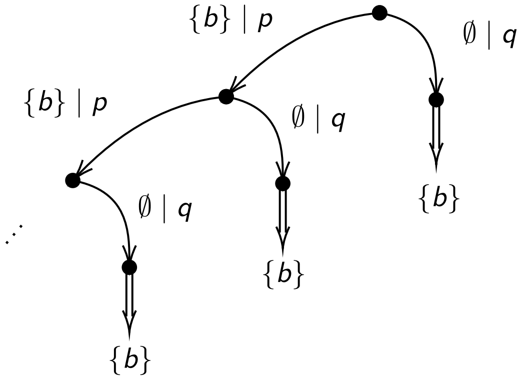
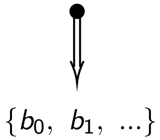
Note: guarded trees may be infinite!

Guarded trees — example

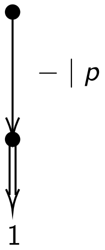


Expressions to trees — base case

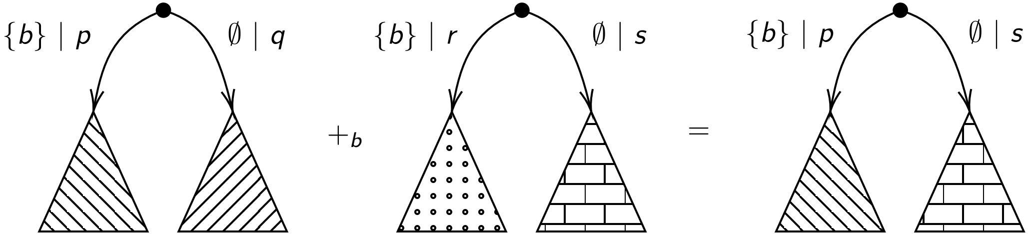
$a \; = \; \{b_0,\; b_1,\; \ldots\} \;\mapsto$



$p \in \Sigma \;\mapsto$



Expressions to trees — Party hat diagrams



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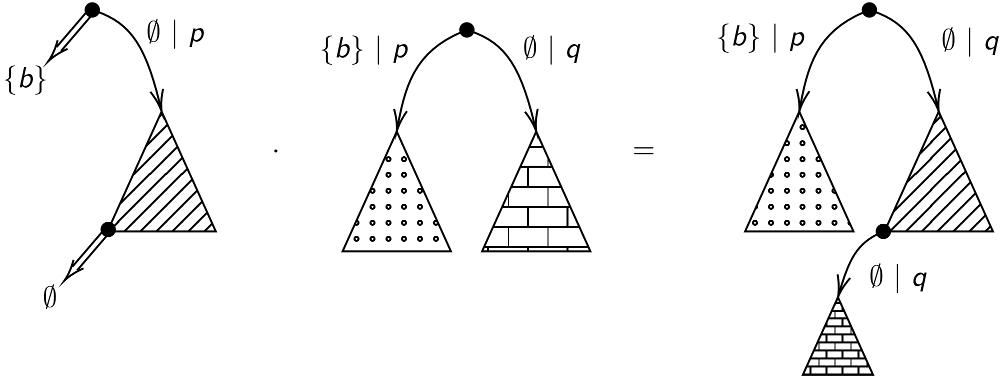

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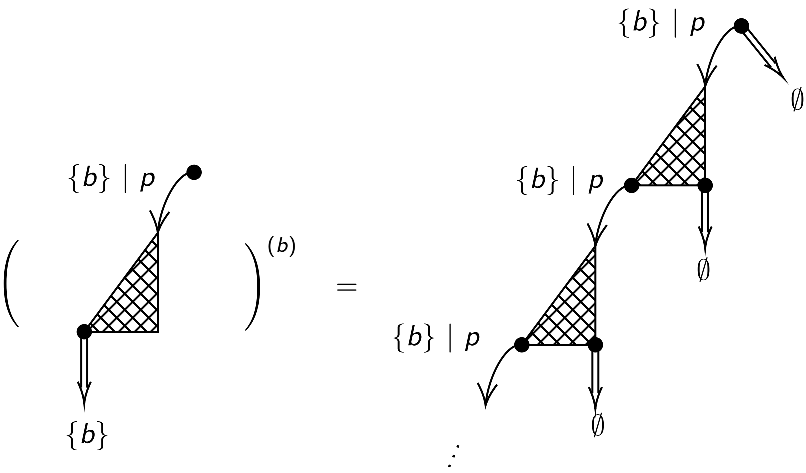

A model in terms of guarded trees

Every expression **e** has an associated guarded tree $[\![\mathbf{e}]\!]$.

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The early termination axiom does *not* hold: $[\![\mathbf{e}0]\!] \neq [\![0]\!]$.

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Question (Soundness & Completeness)

Is $\mathbf{e} \equiv_0 \mathbf{f}$ *equivalent to* $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

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Question (Soundness & Completeness)

Is $\mathbf{e} \equiv_0 \mathbf{f}$ *equivalent to* $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

Question (Decidability)

Can we decide whether $[\![\mathbf{e}]\!] = [\![\mathbf{f}]\!]$*?*

Establishing completeness and decidability

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Note: decision procedures are *nearly linear* — actually feasible!

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It is decidable whether $\mathbf{e} \equiv_0 \mathbf{f}$

Note: decision procedures are *nearly linear* — actually feasible!

The "old" results from (Smolka et al. 2020) can be recovered from these.

Expressiveness

Question

Let $\mathtt{t}$ *be a guarded tree with finitely many distinct subtrees.*

Is there an $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \mathtt{t}$*?*

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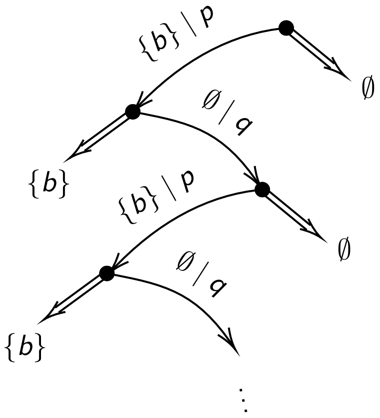
Let t *be a guarded tree with finitely many distinct subtrees.*

Is there an **e** *such that* $[\![\mathbf{e}]\!] = \mathtt{t}$*?*

Not in general — for instance:



See also (Kozen and Tseng 2008).

Expressiveness

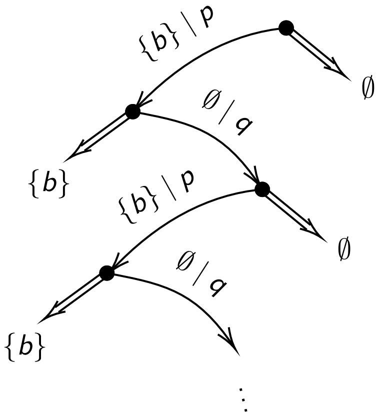
Question

Let $\mathtt{t}$ *be a guarded tree with finitely many distinct subtrees.*

Is there an $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \mathtt{t}$*?*

Reason: our syntax does not have `goto`.
Only *structured* programs!

Not in general — for instance:



See also (Kozen and Tseng 2008).

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```
ℓ0 :if b then p; goto ℓ1 else accept
ℓ1 :if b̄ then q; goto ℓ0 else accept
```

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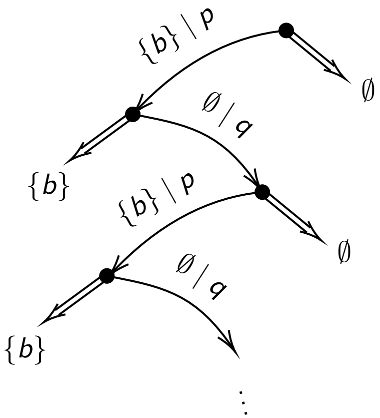

See also (Kozen and Tseng 2008).

Further work

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Is it decidable whether, given a tree $\texttt{t}$*, there exists an* $\mathbf{e}$ *such that* $[\![\mathbf{e}]\!] = \texttt{t}$*?*

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Question

Can we identify rejection and looping without identifying early/late rejection?

What would be the appropriate axioms for such a semantics?

Overview

- GKAT describes general equivalences of programs.
- It admits a complete axiomatization and is decidable.
- The axiom $\mathbf{e}0 \equiv 0$ may not be what you want.
- There is a model for the theory without this axiom.
- Soundness and completeness can be recovered.
- Lack of GOTO means not every tree is expressible.

`https://kap.pe/slides`

`https://doi.org/10.4230/LIPIcs.ICALP.2021.142`

Bonus — Reduction to KAT

Syntax is special case of Kleene Algebra with Tests (KAT):

$$\texttt{if } \mathbf{a} \texttt{ then } \mathbf{e} \texttt{ else } \mathbf{f} \texttt{ end} \mapsto \mathbf{a} \cdot \mathbf{e} + \overline{\mathbf{a}} \cdot \mathbf{f}$$

$$\texttt{while } \mathbf{a} \texttt{ do } \mathbf{e} \texttt{ end} \mapsto (\mathbf{a} \cdot \mathbf{e})^* \cdot \overline{\mathbf{a}}$$

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Known results:

- There is a "nice" set of axioms for KAT.
- Soundness & completeness for a straightforward model.
- Equivalence according to these axioms is decidable.

Bonus — Reduction to KAT

Equivalence in KAT is PSPACE-complete (Cohen, Kozen, and Smith 1996).

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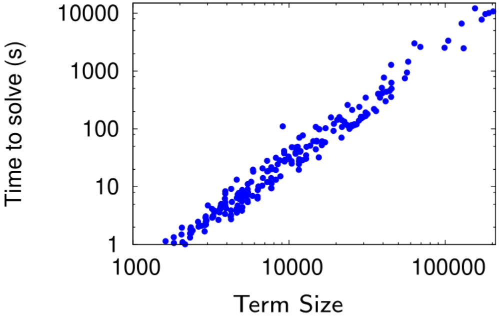
But for practical inputs, good algorithms scale well — e.g., (Foster et al. 2015):



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