An Elementary Proof of the FMP for KA

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Prelude

- The main theorems in this talk are not new, but the proofs are.
- Even if the contents are technical, the techniques are elementary.
- I learned most of these constructions in some form as a sophomore!

Motivation

- Laws of Kleene algebra (KA) model equivalence of regular expressions.

  ☞ Salomaa 1966; Conway 1971; Boffa 1990; Krob 1990; Kozen 1994

- They are also useful when reasoning about programming languages.

  ☞ Kozen and Patron 2000; Anderson et al. 2014; Smolka et al. 2015

- When is something true *only by the laws of KA*?
- How can we concisely show that something is *not* provable in KA?

Kleene algebra
Definition

Definition (Kleene algebra; c.f. Kozen 1994)

A *Kleene algebra* is a tuple $(K, +, \cdot, {}^*, 0, 1)$ where

(1) The "usual" laws for $+$ and $\cdot$ hold (associativity, distributivity, etc. . .)

(2) For all $x, y, z \in K$, the following are true:

$$x + x = x \qquad 1 + x \cdot x^* = x^* \qquad 1 + x^* \cdot x = x^*$$

$$\frac{x + y \cdot z \leq z}{y^* \cdot x \leq z} \qquad \frac{x + y \cdot z \leq y}{x \cdot z^* \leq y}$$

Here, $x \leq y$ is a shorthand for $x + y = y$.

Kleene algebra
Languages

Fix a (finite) set of *letters* Σ, and write Σ^* for the set of words over Σ.

Example (KA of languages)

The KA of *languages over* Σ is given by $(\mathcal{P}(\Sigma^*), \cup, \cdot, {}^*, \emptyset, \{\epsilon\})$, where

- $\mathcal{P}(\Sigma^*)$ is the set of sets of words (*languages*);
- $\cdot$ is pointwise concatenation, i.e., $L \cdot K = \{wx : w \in L, x \in K\}$;
- * is the Kleene star, i.e., $L^* = \{w_1 \cdots w_n : w_1, \ldots, w_n \in L\}$;
- ϵ is the empty word.

Kleene algebra
Relations

Fix a (not necessarily finite) set of *states* S.

Example (KA of relations)

The KA of *relations over* S is given by $(\mathcal{R}(S), \cup, \circ, ^*, \emptyset, \Delta)$, where

- $\mathcal{R}(S)$ is the set of relations on S;
- $\circ$ is relational composition.
- $*$ is the reflexive-transitive closure.
- Δ is the identity relation.

Kleene algebra
Reasoning example

Claim

In every KA $\mathbf{K}$ and for all $\mathbf{u}, \mathbf{v} \in \mathbf{K}$, it holds that $(\mathbf{u} \cdot \mathbf{v})^* \cdot \mathbf{u} \leq \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^*$.

Proof. First, let's recall the fixpoint rule:

$$\frac{\mathbf{x} + \mathbf{y} \cdot \mathbf{z} \leq \mathbf{z}}{\mathbf{y}^* \cdot \mathbf{x} \leq \mathbf{z}}$$

It suffices to prove that $\mathbf{u} + \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^* \leq \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^*$; we derive:

$$\mathbf{u} + \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^* = \mathbf{u} \cdot (1 + \mathbf{v} \cdot \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^*) = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{u})^* \qquad \square$$

Kleene algebra
Expressions

Definition

Exp is the set of *regular expressions*, generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e^*$$

Definition

Given a KA $(K, +, \cdot, {}^*, 0, 1)$ and $h : \Sigma \to K$, we define $\widehat{h} : \mathbf{Exp} \to K$ by

$$\widehat{h}(0) = 0 \qquad \widehat{h}(\mathtt{a}) = h(\mathtt{a}) \qquad \widehat{h}(e \cdot f) = \widehat{h}(e) \cdot \widehat{h}(f)$$
$$\widehat{h}(1) = 1 \qquad \widehat{h}(e + f) = \widehat{h}(e) + \widehat{h}(f) \qquad \widehat{h}(e^*) = \widehat{h}(e)^*$$

If $\ell : \Sigma \to \mathcal{P}(\Sigma^*)$ where $\ell(\mathtt{a}) = \{\mathtt{a}\}$, then $\widehat{\ell}(e)$ is the (regular) language of e.

Let $\boldsymbol{e}, \boldsymbol{f} \in \mathbf{Exp}$. We write ...

- $\boldsymbol{K}, \boldsymbol{h} \models \boldsymbol{e} = \boldsymbol{f}$ when $\boldsymbol{K}$ is a KA and $\boldsymbol{h} : \boldsymbol{\Sigma} \to \boldsymbol{K}$ with $\widehat{\boldsymbol{h}}(\boldsymbol{e}) = \widehat{\boldsymbol{h}}(\boldsymbol{f})$.
- $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ when $\boldsymbol{K}$ is a KA and $\boldsymbol{K}, \boldsymbol{h} \models \boldsymbol{e} = \boldsymbol{f}$ for all $\boldsymbol{h}$.
- $\models \boldsymbol{e} = \boldsymbol{f}$ when $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ for every KA $\boldsymbol{K}$.
- $\mathfrak{F} \models \boldsymbol{e} = \boldsymbol{f}$ when $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ holds in every *finite* KA $\boldsymbol{K}$.
- $\mathfrak{R} \models \boldsymbol{e} = \boldsymbol{f}$ when $\mathcal{R}(\boldsymbol{S}) \models \boldsymbol{e} = \boldsymbol{f}$ for all $\boldsymbol{S}$.

Kleene algebra
Model theory

$$
\begin{array}{ccc}
\mathfrak{F} \models e = f & \xLeftrightarrow{\text{(Palka 2005)}} & \models e = f \\
 & & \Updownarrow \text{(Kozen 1994)} \\
\mathfrak{R} \models e = f & \xLeftrightarrow{\text{(Pratt 1980)}} & \mathcal{P}(\Sigma^*) \models e = f
\end{array}
$$

Main result

In a nutshell

Palka's proof relies on Kozen's completeness theorem. She writes:

> *…an independent proof of [the finite model property] would provide a quite different proof of the Kozen completeness theorem, based on purely logical tools. We defer this task to further research.* (Palka 2005)

We found such a proof — with many ideas inspired by Palka.

Main result
A roadmap

Need to show: if $\mathfrak{F} \models \boldsymbol{e} = \boldsymbol{f}$, then $\models \boldsymbol{e} = \boldsymbol{f}$.

Given $\boldsymbol{e}, \boldsymbol{f} \in \mathbf{Exp}$ we do the following:

1. Turn expressions $\boldsymbol{e}$ into a finite automaton $\boldsymbol{A_e}$
2. Convert the finite automaton $\boldsymbol{A_e}$ into a finite monoid $\boldsymbol{M_e}$
3. Translate the finite monoid $\boldsymbol{M_e}$ into a finite KA $\boldsymbol{K_e}$
4. Prove something about interpretations inside $\boldsymbol{K_e}$
5. Apply the premise that $\models \boldsymbol{e} = \boldsymbol{f}$

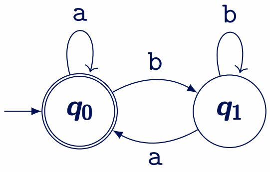
Expressions to automata

Definition

An automaton is a tuple $\boldsymbol{A} = (\boldsymbol{Q}, \to, \boldsymbol{I}, \boldsymbol{F})$ where

- $\boldsymbol{Q}$ is a finite set of *states*; and
- $\to \subseteq \boldsymbol{Q} \times \boldsymbol{\Sigma} \times \boldsymbol{Q}$ is the *transition relation*;
- $\boldsymbol{I} \subseteq \boldsymbol{Q}$ is the set of *initial states*
- $\boldsymbol{F} \subseteq \boldsymbol{Q}$ is the set of *accepting states*

We write $q \xrightarrow{\mathtt{a}} q'$ when $(q, \mathtt{a}, q') \in \to$.



The *language* of $q \in Q$ is $L_A(q) = \{\mathtt{a}_1 \cdots \mathtt{a}_n \in \Sigma^* : q \xrightarrow{\mathtt{a}_1} \circ \cdots \circ \xrightarrow{\mathtt{a}_n} q' \in F\}$

The language of A is given by $\bigcup_{q \in I} L_A(q)$.

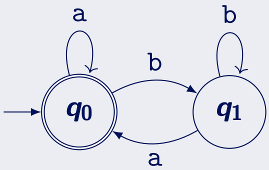
Expressions to automata

Lemma (c.f. Kleene 1956; Brzozowski 1964; Antimirov 1996)

For every $\boldsymbol{e}$, we can construct an automaton $\boldsymbol{A_e}$ that accepts the language of $\boldsymbol{e}$.

Example

Here is the automaton $\boldsymbol{A_e}$ for $\boldsymbol{e} = \mathtt{a}^* \cdot (\mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^*$:



Automata to monoids

Let $\boldsymbol{A} = (\boldsymbol{Q}, \rightarrow, \boldsymbol{I}, \boldsymbol{F})$ be an automaton.

Definition (Transition monoid; McNaughton and Papert 1968)

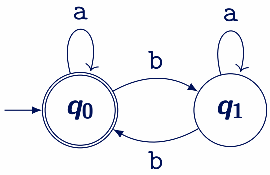
$(\boldsymbol{M_A}, \circ, \boldsymbol{\Delta})$ is the monoid where $\boldsymbol{M_A} = \{\xrightarrow{a_1} \circ \cdots \circ \xrightarrow{a_n} : a_1 \cdots a_n \in \boldsymbol{\Sigma}^*\}$.

Example

The transition monoid for the automaton $\boldsymbol{A}$ on the right is carried by $\boldsymbol{M_A} = \{\xrightarrow{a}, \xrightarrow{b}\}$, where

$$\xrightarrow{a} = \{(q_0, q_0), (q_1, q_1)\}$$

$$\xrightarrow{b} = \{(q_0, q_1), (q_0, q_1)\}$$



Monoids to Kleene algebras

Lemma (Palka 2005)

Let $(M, \cdot, 1)$ *be a monoid. Now* $(\mathcal{P}(M), \cup, \otimes, {}^{\circledast}, \emptyset, \{1\})$ *is a KA, where*

$$T \otimes U = \{t \cdot u : t \in T \wedge u \in U\} \qquad T^{\circledast} = \{t_1 \cdots t_n : t_1, \ldots, t_n \in T\}$$

Putting it all together

Given an expression $\boldsymbol{e}$, we can now obtain a *finite* KA $\boldsymbol{K_e} = \mathcal{P}(\boldsymbol{M_{A_e}})$.

Lemma

Let $\boldsymbol{e}, \boldsymbol{f} \in \textbf{Exp}$. *If* $\boldsymbol{K_e} \models \boldsymbol{e} = \boldsymbol{f}$ *and* $\boldsymbol{K_f} \models \boldsymbol{e} = \boldsymbol{f}$, *then* $\models \boldsymbol{e} = \boldsymbol{f}$.

Theorem (Finite model property)

If $\mathfrak{F} \models \boldsymbol{e} = \boldsymbol{f}$ *then* $\models \boldsymbol{e} = \boldsymbol{f}$.

Peeling the onion

Solving automata

Definition

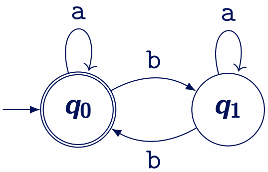
Let $(Q, \to, I, F)$ be an automaton. A *solution* is a function $s : Q \to \mathbf{Exp}$ such that

$$\models F(q) + \sum_{q \xrightarrow{a} q'} a \cdot s(q') \leq s(q) \qquad F(q) = \begin{cases} 1 & q \in F \\ 0 & q \notin F \end{cases}$$

Example

For the automaton on the right, a solution satisfies

$$\models 1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot s(q_1) \leq s(q_0)$$
$$\models 0 + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$



Peeling the onion
Solving automata

Example (Continued)

We start with the second condition:

$$\mathbf{0} + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$

We can rewrite this as

$$\mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$

which by the fixpoint rule implies

$$\mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0) \leq s(q_1)$$

Example (Continued)

Now we look at the second condition

$$1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot s(q_1) \leq s(q_0)$$

Substituting $\mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0) \leq s(q_1)$ we get

$$1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0) \leq s(q_0)$$

which rewrites to

$$1 + (\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b}) \cdot s(q_0) \leq s(q_0)$$

By the fixpoint rule, $(\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^* \leq s(q_0)$.

Example (Continued)

We now have two lower bounds:

$$(\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^* \leq \boldsymbol{s}(\boldsymbol{q}_0)$$
$$\mathtt{a}^* \cdot \mathtt{b} \cdot (\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^* \leq \boldsymbol{s}(\boldsymbol{q}_1)$$

It turns these are also solutions to $\boldsymbol{A}$ — thus we found the least solution.

Peeling the onion
Solving automata

Theorem (Kleene 1956; see also Conway 1971)

Every automaton admits a least solution (unique up to equivalence).

When $\boldsymbol{A}$ is an automaton, we write

- $\overline{\boldsymbol{A}}(\boldsymbol{q})$ for the least solution to $\boldsymbol{A}$ at $\boldsymbol{q}$
- $\lfloor \boldsymbol{A} \rfloor$ for the sum of $\overline{\boldsymbol{A}}(\boldsymbol{q})$ for $\boldsymbol{q} \in \boldsymbol{I}$

Lemma

If $\boldsymbol{e} \in \mathbf{Exp}$, then $\models \lfloor \boldsymbol{A_e} \rfloor \leq \boldsymbol{e}$.

Definition (Transition automaton; McNaughton and Papert 1968)

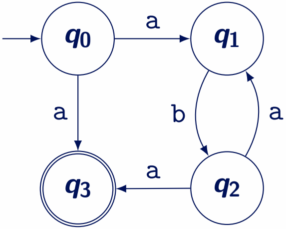
Let $R \in M_A$. We write $A[R]$ for the automaton $(M_A, \rightarrow_\circ, \{\Delta\}, \{R\})$ where

$$P \xrightarrow{a}_\circ Q \iff P \circ \xrightarrow{a} = Q$$

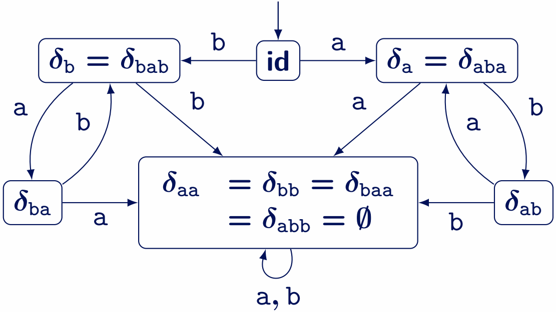
Intuition: $w \in L(A[R])$ means $q\ R\ q'$ iff w traces from q to q' in A.

Peeling the onion

Solving monoids



$\Rightarrow$



Lemma (Solving transition automata)

Let $\boldsymbol{A}$ be an automaton, let $\boldsymbol{q} \in \boldsymbol{Q}$ and let $\boldsymbol{R} \in \boldsymbol{M_A}$ with $\boldsymbol{q}\ \boldsymbol{R}\ \boldsymbol{q_f} \in \boldsymbol{F}$. We have

$$\models \lfloor \boldsymbol{A}[\boldsymbol{R}] \rfloor \leq \overline{\boldsymbol{A}}(\boldsymbol{q})$$

Let $\boldsymbol{h_e} : \boldsymbol{\Sigma} \to \boldsymbol{K_e}$ be given by $\boldsymbol{h_e}(\mathrm{a}) = \{\xrightarrow{\mathrm{a}}_{\boldsymbol{e}}\}$.

Lemma

Let $\boldsymbol{e} \in \mathbf{Exp}$ and let $\boldsymbol{R} \in \widehat{\boldsymbol{h_e}}(\boldsymbol{e})$. Then $\models \overline{\boldsymbol{A_e}[\boldsymbol{R}]} \leq \boldsymbol{e}$.

Peeling the onion

Solving Kleene algebras

Let $h_e : \Sigma \to K_e$ be given by $h_e(\mathrm{a}) = \{\xrightarrow{\mathrm{a}}_e\}$.

Lemma

Let $e, f \in \mathbf{Exp}$*. We have that*

$$\models f \leq \sum_{R \in \widehat{h_e}(f)} \lfloor A_e[R] \rfloor$$

Proof sketch.

By induction on f. □

Peeling the onion

Proving the main lemma

Lemma

Let $\boldsymbol{e}, \boldsymbol{f} \in \mathbf{Exp}$. *If* $\boldsymbol{K_e} \models \boldsymbol{e} = \boldsymbol{f}$ *and* $\boldsymbol{K_f} \models \boldsymbol{e} = \boldsymbol{f}$, *then* $\models \boldsymbol{e} = \boldsymbol{f}$.

Proof.

Since $\boldsymbol{K_e} \models \boldsymbol{e} = \boldsymbol{f}$, we have that $\widehat{\boldsymbol{h_e}}(\boldsymbol{e}) = \widehat{\boldsymbol{h_e}}(\boldsymbol{f})$; we can then derive

$$\models \boldsymbol{f} \leq \sum_{\boldsymbol{R} \in \widehat{\boldsymbol{h_e}}(\boldsymbol{f})} \lfloor \boldsymbol{A_e}[\boldsymbol{R}] \rfloor = \sum_{\boldsymbol{R} \in \widehat{\boldsymbol{h_e}}(\boldsymbol{e})} \lfloor \boldsymbol{A_e}[\boldsymbol{R}] \rfloor \leq \boldsymbol{e}$$

By a similar argument, $\models \boldsymbol{e} \leq \boldsymbol{f}$; the claim then follows. □

Peeling the onion
The grand finale

Theorem

If $\mathfrak{F} \models \boldsymbol{e} = \boldsymbol{f}$, *then* $\models \boldsymbol{e} = \boldsymbol{f}$.

Proof.

Since $\boldsymbol{K_e}$ and $\boldsymbol{K_f}$ are finite KAs, we have that $\boldsymbol{K_e} \models \boldsymbol{e} = \boldsymbol{f}$ and $\boldsymbol{K_f} \models \boldsymbol{e} = \boldsymbol{f}$.

The proof then follows by the previous lemma. □

Some thoughts

- The proof uses *Antimirov's* instead of *Brzozowski's construction*.
- We do not rely on bisimilarity-based arguments at all (c.f. Jacobs 2006).
- We do not use the right-handed axioms for the star:

$$1 + x \cdot x^* = x^* \qquad \frac{x + y \cdot z \leq y}{x \cdot z^* \leq y}$$

  - These were known not to be necessary

    ☞ Krob 1990; Boffa 1990; Das, Doumane, and Pous 2018; Kozen and Silva 2020

  - Upshot: a proof-theoretic result for KA: "right-hand elimination".

Coq formalization

- All results formalized in the Coq proof assistant.
- Trusted base:
  - Calculus of Inductive Constructions.
  - Streicher's *axiom K*.
  - Dependent functional extensionality.
- Some concepts are encoded differently; ideas remain the same.

Further open questions

- Can we apply these ideas to *guarded Kleene algebra with tests*?
- Do these techniques extend to *KA with hypotheses*?
- Is there a representation theorem or duality for KA?

`https://kap.pe/slides` `https://kap.pe/papers`

Bonus: extending the model theory

Lemma

If $\mathfrak{FR} \models e = f$, *then* $\models e = f$.

Proof sketch.

We show that $\mathfrak{FR} \models e = f$ implies $\mathcal{P}(\Sigma^*) \models e = f$. For $n \in \mathbb{N}$, choose

$$\Sigma_n = \{w \in \Sigma^* : |w| \leq n\} \qquad h_n : \Sigma \to \mathcal{R}(\Sigma_n),\ \mathtt{a} \mapsto \{(w, w\mathtt{a}) : w\mathtt{a} \in \Sigma_n\}$$

For all $w \in \Sigma_n$ and regular expressions g, we now have $w \in \widehat{\ell}(g)$ iff $(\epsilon, w) \in \widehat{h_n}(g)$.

Thus $w \in \hat{\ell}(f)$ if and only if $w \in \widehat{h_{|w|}}(e) = \widehat{h_{|w|}}(f)$ if and only if $w \in \hat{\ell}(f)$.

This means that $\mathcal{P}(\Sigma^*), \ell \models e = f$, whence $\mathcal{P}(\Sigma^*) \models e = f$. □

Bonus: pomsets

Expressions in *concurrent KA* (CKA) are generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e \parallel f \mid e^* \mid e^\dagger$$

Definition (Bi-KA)

A *bi-KA* is a tuple $(K, +, \cdot, \parallel, {}^*, {}^\dagger, 0, 1)$ where

- $(K, +, \cdot, {}^*)$ and $(K, +, \parallel, {}^\dagger)$ are both KAs, and
- $\parallel$ commutes, i.e., $K \models e \parallel f = f \parallel e$.

A *weak bi-KA* is a bi-KA without the †.

Definition (Concurrent KA)

A *(weak) concurrent KA* is a (weak) bi-KA s.t. $(e \parallel g) \cdot (f \parallel h) \leq (e \cdot f) \parallel (g \cdot h)$.

Bonus: pomsets

Example

The *bi-KA of pomset languages* over $\mathbf{\Sigma}$ is $(\mathcal{P}(\mathbf{Pom}(\mathbf{\Sigma})), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{\mathbf{1}\})$, where

- $\mathbf{Pom}(\mathbf{\Sigma})$ denotes the set of pomsets over $\mathbf{\Sigma}$;
- $\mathbf{1}$ denotes the empty pomset;
- $L \cdot L' = \{U \cdot V : U \in L, V \in L'\}$ and similarly for $\parallel$; and
- $L^* = \{1\} \cup L \cup L \cdot L \cup \cdots$ and $L^\dagger = \{1\} \cup L \cup L \parallel L \cup \cdots$.

Bonus: pomsets

Example

The *concurrent KA of pomset ideals* over $\boldsymbol{\Sigma}$ is $(\mathcal{I}(\boldsymbol{\Sigma}), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{1\})$, where

- $\mathcal{I}(\boldsymbol{\Sigma})$ contains the pomset languages downward-closed under $\sqsubseteq$; and
- the operators are as for bi-KA, but followed by downward closure under $\sqsubseteq$.

Bonus: pomsets

Theorem (Laurence and Struth 2014)

Let $\boldsymbol{e}$ *and* $\boldsymbol{f}$ *be (weak) concurrent KA expressions.*

Now $\mathcal{P}(\mathbf{Pom}(\boldsymbol{\Sigma})) \models \boldsymbol{e} = \boldsymbol{f}$ *if and only if* $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ *for all (weak) bi-KAs* $\boldsymbol{K}$

Theorem (Laurence and Struth 2017; K., Brunet, Silva, et al. 2018)

Let $\boldsymbol{e}$ *and* $\boldsymbol{f}$ *be weak concurrent KA expressions.*

Now $\mathcal{I}(\boldsymbol{\Sigma}) \models \boldsymbol{e} = \boldsymbol{f}$ *if and only if* $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ *for all weak CKAs* $\boldsymbol{K}$

Bonus: pomsets

Conjecture

Let $\boldsymbol{e}$ and $\boldsymbol{f}$ be concurrent KA expressions.

Now $\boldsymbol{\mathcal{I}(\Sigma)} \models \boldsymbol{e} = \boldsymbol{f}$ if and only if $\boldsymbol{K} \models \boldsymbol{e} = \boldsymbol{f}$ for all CKAs $\boldsymbol{K}$

Current techniques do not work!

Bonus: pomsets

<speculation>

Bonus: pomsets

The following roadmap *might* work:

1. Translate CKA expressions to automata

   ⇒ Pomset automata (K., Brunet, Luttik, et al. 2019)

   ⇒ or HDAs (van Glabbeek 2004; Fahrenberg 2005; Fahrenberg et al. 2022)

2. Translate these automata to *ordered bimonoids* (Bloom and Ésik 1996)

   ⇒ see also (Lodaya and Weil 2000; van Heerdt et al. 2021)

3. Translate bimonoids to concurrent KAs.

   ⇒ essentially the same recipe?

Bonus: pomsets

</speculation>

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