A Complete Inference System for Skip-free Guarded Kleene Algebra with Tests

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* who kindly let me use his slides
Let’s play a game of *Fizzbuzz*!

- Take turns counting to 100.
- Number divisible by 3 ⇒ say *Fizz*!
- Number divisible by 5 ⇒ say *Buzz*!
- Number is divisible by 3 and 5 ⇒ say *Fizzbuzz*!
- Otherwise, just say the number.
Let’s play a game of Fizzbuzz!

- Take turns counting to 100.
- Number divisible by 3 (but not 5) ⇒ say Fizz!
- Number divisible by 5 (but not 3) ⇒ say Buzz!
- Number is divisible by 3 and 5 ⇒ say Fizzbuzz!
- Otherwise, just say the number.
Reasoning about software: A story

def fizzbuzz1 =
    n := 1;
    while n ≤ 100 do
        if 3|n then
            if not 5|n then
                print fizz; n++;
            else
                print fizzbuzz; n++;
        else if 5|n then
            print buzz; n++;
        else
            print n; n++;
    print done!
Reasoning about software: A story

\[
\text{def } \text{fizzbuzz1 =}
\begin{align*}
& n := 1; \\
& \text{while } n \leq 100 \text{ do} \\
& \quad \text{if } 3 | n \text{ then} \\
& \quad \quad \text{if not } 5 | n \text{ then} \\
& \quad \quad \quad \text{print } \text{fizz}; n++; \\
& \quad \quad \text{else} \\
& \quad \quad \quad \text{print } \text{fizzbuzz}; n++; \\
& \quad \text{else if } 5 | n \text{ then} \\
& \quad \quad \text{print } \text{buzz}; n++; \\
& \quad \text{else} \\
& \quad \quad \text{print } n; n++; \\
& \text{print } \text{done!};
\end{align*}
\]

\[
\text{def } \text{fizzbuzz2 =}
\begin{align*}
& n := 1; \\
& \text{while } n \leq 100 \text{ do} \\
& \quad \text{if } 5 | n \text{ and } 3 | n \text{ then} \\
& \quad \quad \text{print } \text{fizzbuzz}; \\
& \quad \text{else if } 3 | n \text{ then} \\
& \quad \quad \text{print } \text{fizz}; \\
& \quad \text{else if } 5 | n \text{ then} \\
& \quad \quad \text{print } \text{buzz}; \\
& \quad \text{else} \\
& \quad \quad \text{print } n; \\
& \quad \quad n++; \\
& \text{print } \text{done!};
\end{align*}
\]
Reasoning about software: A story

\[
def \text{fizzbuzz1} = \\
n := 1; \\
\text{while } n \leq 100 \text{ do} \\
\quad \text{if } 3 \mid n \text{ then} \\
\quad \quad \text{if not } 5 \mid n \text{ then} \\
\quad \quad \quad \text{print } \text{fizz}; n++; \\
\quad \quad \text{else} \\
\quad \quad \quad \text{print } \text{fizzbuzz}; n++; \\
\quad \text{else if } 5 \mid n \text{ then} \\
\quad \quad \text{print } \text{buzz}; n++; \\
\quad \text{else} \\
\quad \quad \text{print } n; n++; \\
\text{print } \text{done!};
\]

\[
def \text{fizzbuzz2} = \\
n := 1; \\
\text{while } n \leq 100 \text{ do} \\
\quad \text{if } 5 \mid n \text{ and } 3 \mid n \text{ then} \\
\quad \quad \text{print } \text{fizzbuzz}; \\
\quad \text{else if } 3 \mid n \text{ then} \\
\quad \quad \text{print } \text{fizz}; \\
\quad \text{else if } 5 \mid n \text{ then} \\
\quad \quad \text{print } \text{buzz}; \\
\quad \text{else} \\
\quad \quad \text{print } n; \\
\quad \quad n++; \\
\text{print } \text{done!};
\]

\[\text{fizzbuzz1} \equiv \text{fizzbuzz2}\]
Reasoning about software: A story

Starting with fizzbuzz1...

```plaintext
n := 1;
while n ≤ 100 do
  if 3|n then
    if not 5|n then
      print fizz; n++;
    else
      print fizzbuzz; n++;
  else if 5|n then
    print buzz; n++;
  else
    print n; n++;
print done!;
```
Reasoning about software: A story

Move the `n++;` to the end...

```
\[\begin{align*}
n &:= 1; \\
\text{while } &n \leq 100 \text{ do} \\
\text{if } &3|n \text{ then} \\
\text{ if } &\neg 5|n \text{ then} \\
\text{ print } &fizz; \\
\text{ else} &\text{ print } fizzbuzz; \\
\text{else if } &5|n \text{ then} \\
\text{ print } &buzz; \\
\text{ else} &\text{ print } n; \\
\end{align*}\]
```

`n++;`

print `done!`;
Reasoning about software: A story

Negate not $5|n$ and flip the branches

```
n := 1;
while $n \leq 100$ do
  if $3|n$ then
    if $5|n$ then
      print fizzbuzz;
    else
      print fizz;
  else if $5|n$ then
    print buzz;
  else
    print $n$;
  $n$++;
print done!
```
Reasoning about software: A story

Merge $3|n$ and $5|n$

$n := 1$
while $n \leq 100$ do
  if $3|n$ and $5|n$ then
    print *fizzbuzz*;
  else if $3|n$ then
    print *fizz*;
  else if $5|n$ then
    print *buzz*;
  else
    print *n*;
  $n++$
print *done!*;
Reasoning about software: A story

This is precisely `fizzbuzz2`!

```
n := 1;
while n ≤ 100 do
    if 3|n and 5|n then
        print `fizzbuzz`;
    else if 3|n then
        print `fizz`;
    else if 5|n then
        print `buzz`;
    else
        print n;
n++;
print `done`!
```
The reasoning steps applied are very general. For instance:

\[
\text{if not } b \text{ then } e \text{ else } f = \text{if } b \text{ then } f \text{ else } e
\]

should work regardless of what \( b, e \) and \( f \) are.
Taking a step back

- The reasoning steps applied are very general. For instance:

  \[
  \text{if not b then e else f} = \text{if b then f else e}
  \]

  should work regardless of what \( b, e \) and \( f \) are.

- We treated the program as an expression, and reasoned equationally.

  *programs are mathematical expressions, […] subject to a set of laws as rich and elegant as those of any other branch of mathematics* (Hoare et al. 1984)
Taking a step back

By abstracting away from individual actions and tests, we go from . . .

```python
def fizzbuzz1 =
    n := 1;
    while n <= 100 do
        if 3|n then
            if not 5|n then
                print fizz; n++;
            else
                print fizzbuzz; n++;
        else if 5|n then
            print buzz; n++;
        else
            print n; n++;
    print done!;

def fizzbuzz2 =
    n := 1;
    while n <= 100 do
        if 5|n and 3|n then
            print fizzbuzz;
        else if 3|n then
            print fizz;
        else if 5|n then
            print buzz;
        else
            print n;
            n++;
    print done!;
```
Taking a step back

...to propositional programs:

```python
def fizzbuzz1 =
p;
while b do
  if c then
    if not d then
      r; u;
    else
      q; u;
  else if d then
    s; u;
  else
    t; u;
  v;
def fizzbuzz2 =
p;
while b do
  if d and c then
    q;
  else if c then
    r;
  else if d then
    s;
  else
    t;
u;
v;
```
Formalizing our reasoning

\[
\text{if } b \text{ then } e \text{ else } f = \text{if } \neg b \text{ then } f \text{ else } e \quad (\text{skew commutativity})
\]
Formalizing our reasoning

\[
\text{if } b \text{ then } e \text{ else } f = \text{if not } b \text{ then } f \text{ else } e \quad \text{(skew commutativity)}
\]

\[
(\text{if } b \text{ then } e \text{ else } f) ; g = \text{if } b \text{ then } e ; g \text{ else } f ; g \quad \text{(left distributivity)}
\]
Formalizing our reasoning

\[
\text{if } b \text{ then } e \text{ else } f = \text{if not } b \text{ then } f \text{ else } e \tag{skew commutativity}
\]

\[
(\text{if } b \text{ then } e \text{ else } f);g = \text{if } b \text{ then } e;g \text{ else } f;g \tag{left distributivity}
\]

\[
\text{if } b \text{ then }
\begin{align*}
\text{if } c \text{ then } e \text{ else } f \\
\text{else } g
\end{align*}
= \text{if } b \text{ and } c \text{ then }
\begin{align*}
e \\
\text{else } (\text{if } b \text{ then } f \text{ else } g)
\end{align*} \tag{skew associativity}
\]
Formalizing our reasoning

\[
\text{if } b \text{ then } e \text{ else } f = \text{if } \neg b \text{ then } f \text{ else } e \quad (\text{skew commutativity})
\]

\[
(\text{if } b \text{ then } e \text{ else } f) ; g = \text{if } b \text{ then } e ; g \text{ else } f ; g \quad (\text{left distributivity})
\]

\[
\text{if } b \text{ then } \\
\quad (\text{if } c \text{ then } e \text{ else } f) \\
\text{else } \\
\quad g = \text{if } b \text{ and } c \text{ then } \\
\quad \quad e \\
\text{else } \\
\quad \quad (\text{if } b \text{ then } f \text{ else } g) \quad (\text{skew associativity})
\]

\[
\text{while } b \text{ do } \\
\quad e = \text{if } b \text{ then } \\
\quad \quad e \\
\text{while } b \text{ do } \\
\quad e \quad (\text{loop unrolling})
\]
Formalizing our reasoning: Kleene Algebra with Tests

Fix a set \{p, q, \ldots\} of actions and a Boolean algebra \{b, c, \ldots\} of tests

\[
\text{\begin{align*}
& b \mid p \mid e + f \mid e; f \mid e^* \\
& \\
& \text{Extends regular expressions with a Boolean algebra of tests} \\
& \quad \text{if } b \text{ then } e \text{ else } f = b; e + (\text{not } b); f \\
& \quad \text{while } b \text{ do } e = (b; e)^* ; (\text{not } b)
\end{align*}}
\]

(Kozen 1996), (Kozen & Smith 1996)
Formalizing our reasoning: Kleene Algebra with Tests

Fix a set \{p, q, \ldots\} of actions and a Boolean algebra \{b, c, \ldots\} of tests

\[ b | p | e + f | e; f | e^* \]

- Extends regular expressions with a Boolean algebra of tests

  \[
  \begin{align*}
  \text{if } b \text{ then } e \text{ else } f &= b; e + (\text{not } b); f \\
  \text{while } b \text{ do } e &= (b; e)^* ; (\text{not } b)
  \end{align*}
  \]

- Language semantics in terms of guarded strings: \( \alpha_1 p_1 \alpha_2 p_2 \alpha_3 p_3 \cdots \alpha_n p_n \alpha_{n+1} \)

(Kozen 1996), (Kozen & Smith 1996)
Formalizing our reasoning: Kleene Algebra with Tests

Fix a set \( \{p, q, \ldots \} \) of actions and a Boolean algebra \( \{b, c, \ldots \} \) of tests

\[
\begin{align*}
    b & \mid p \mid e + f \mid e; f \mid e^*
\end{align*}
\]

- Extends regular expressions with a Boolean algebra of tests
  
  \[
  \text{if } b \text{ then } e \text{ else } f = b; e + (\text{not } b); f
  \]
  
  \[
  \text{while } b \text{ do } e = (b; e)^* ; (\text{not } b)
  \]

- Language semantics in terms of guarded strings: \( \alpha_1 p_1 \alpha_2 p_2 \alpha_3 p_3 \cdots \alpha_n p_n \alpha_{n+1} \)

- Complete and finitary axiomatization

(Kozen 1996), (Kozen & Smith 1996)
Formalizing our reasoning: Kleene Algebra with Tests

Fix a set \( \{p, q, \ldots \} \) of actions and a Boolean algebra \( \{b, c, \ldots \} \) of tests

\[
\begin{align*}
  b & \mid p \mid e + f \mid e; f \mid e^* \\
\end{align*}
\]

- Extends regular expressions with a Boolean algebra of tests

  \[
  \text{if } b \text{ then } e \text{ else } f = b; e + (\text{not } b); f \\
  \text{while } b \text{ do } e = (b; e)^* ; (\text{not } b)
  \]

- Language semantics in terms of guarded strings: \( \alpha_1 p_1 \alpha_2 p_2 \alpha_3 p_3 \cdots \alpha_n p_n \alpha_{n+1} \)

- Complete and finitary axiomatization

- Non-determinism makes equivalence PSPACE-complete

(Kozen 1996), (Kozen & Smith 1996)
Formalizing our reasoning: Guarded KAT

Fix a set \( \{p, q, \ldots \} \) of actions and a Boolean algebra \( \{b, c, \ldots \} \) of tests

\[
\begin{align*}
    b & \mid p \mid e + b f \mid e; f \mid e^{(b)} \\
\end{align*}
\]

The part of KAT specifically for while programs!

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Fix a set \{p, q, \ldots\} of actions and a Boolean algebra \{b, c, \ldots\} of tests

\[
\begin{align*}
    b \mid p \mid e +_b f \mid e; f \mid e^{(b)}
\end{align*}
\]

The part of KAT specifically for while programs!

\[
\begin{align*}
    e +_b f &= \text{if } b \text{ then } e \text{ else } f \\
    e^{(b)} &= \text{while } b \text{ do } e
\end{align*}
\]

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Formalizing our reasoning: Guarded KAT

Fix a set \( \{p, q, \ldots\} \) of actions and a Boolean algebra \( \{b, c, \ldots\} \) of tests

\[
 b \mid p \mid e +_b f \mid e; f \mid e^{(b)}
\]

The part of KAT specifically for while programs!

\[
 e +_b f = \text{if } b \text{ then } e \text{ else } f \\
= b; e + (\text{not } b); f \\
\]

\[
 e^{(b)} = \text{while } b \text{ do } e \\
= (b; e)^*; (\text{not } b)
\]

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Formalizing our reasoning: Guarded KAT

Fix a set \{p, q, \ldots\} of actions and a Boolean algebra \{b, c, \ldots\} of tests

\[
b \mid p \mid e +_b f \mid e; f \mid e^{(b)}
\]

The part of KAT specifically for while programs!

\[
e +_b f = \text{if } b \text{ then } e \text{ else } f
\]
\[
e^{(b)} = \text{while } b \text{ do } e
\]
\[
= b; e + (\text{not } b); f
\]
\[
= (b; e)^* ; (\text{not } b)
\]

\[
\text{fizzbuzz1} = p; ((r; u + \text{not } d \ q; u) + c (s; u + d \ t; u))^{(b)} ; v
\]
\[
\text{fizzbuzz2} = p; ((q + d \ and \ c (r + c (s + d \ t)))u)^{(b)} ; v
\]

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Formalizing our reasoning: Guarded KAT

Fix a set \{p, q, \ldots\} of actions and a Boolean algebra \{b, c, \ldots\} of tests

\[ b \mid p \mid e +_b f \mid e; f \mid e^{(b)} \]

- Language semantics in terms of guarded strings

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Formalizing our reasoning: Guarded KAT

Fix a set \( \{p, q, \ldots \} \) of actions and a Boolean algebra \( \{b, c, \ldots \} \) of tests

\[
\begin{align*}
  b &| p & e +_b f &| e; f &| e^{(b)} 
\end{align*}
\]

- Language semantics in terms of guarded strings
- Language equivalence is efficiently decidable! (nearly linear)

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Formalizing our reasoning: Guarded KAT

Fix a set \( \{p, q, \ldots \} \) of actions and a Boolean algebra \( \{b, c, \ldots \} \) of tests

\[
b | p | e \oplus b \ f | e ; f | e^{(b)}
\]

- Language semantics in terms of guarded strings
- Language equivalence is efficiently decidable! (nearly linear)
- Complete but infinitary axiomatization

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

Conditionals

\[ e +_b e = e \]
\[ e +_b f = f + \neg b e \]
\[ (e +_b f) +_c g = e +_b (f +_c g) \]
\[ e +_b f = b; e +_b f \]

Composition

\[ b; c = b \text{ and } c \]
\[ 0; e = e; 0 = 0 \]
\[ 1; e = e; 1 = e \]
\[ e; (f; g) = (e; f); g \]
\[ (e +_b f); g = e; g +_b f; g \]

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

**Conditionals**

\[
\begin{align*}
e +_b e &= e \\
e +_b f &= f + \text{not } b e \\
(e +_b f) +_c g &= e +_b \text{ and } c (f +_c g) \\
e +_b f &= b; e +_b f
\end{align*}
\]

**Composition**

\[
\begin{align*}
b; c &= b \text{ and } c \\
0; e &= e; 0 = 0 \\
1; e &= e; 1 = e \\
e; (f; g) &= (e; f); g \\
(e +_b f); g &= e; g +_b f; g
\end{align*}
\]

fizzbuzz1

\[
= p; ((q; u + \text{not } d; r; u) + c (s; u + d; t; u))^{(b)}; v
\]

fizzbuzz2

\[
= \text{fizzbuzz2}
\]

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

**Conditionals**

\[
\begin{align*}
& e +_b e = e \\
& e +_b f = f + \text{not } b\ e \\
& (e +_b f) +_c g = e +_b \text{ and } c\ (f +_c g) \\
& e +_b f = b; e +_b f
\end{align*}
\]

**Composition**

\[
\begin{align*}
& b; c = b \text{ and } c \\
& 0; e = e; 0 = 0 \\
& 1; e = e; 1 = e \\
& e; (f; g) = (e; f); g \\
& (e +_b f); g = e; g +_b f; g
\end{align*}
\]

\[
\text{fizzbuzz1} = p; (((q; u + \text{not } d\ r; u) + c\ (s; u + d\ t; u))^{(b)}; v \\
= p; (((q + \text{not } d\ r) + c\ (s + d\ t)); u)^{(b)}; v
\]

\[
\text{fizzbuzz2} = p; pp; ppp\ q; u \\
= d\ q\ c\ p\ s\ q\ c\ d\ t\ qq\ v
\]

(\text{Smolka, Foster, Hsu, K., Kozen & Silva 2019})
Axiomatizing our reasoning

Conditions

\[ e +_b e = e \]
\[ e +_b f = f + \text{not } b \ e \]
\[ (e +_b f) +_c g = e +_b \text{ and } c \ (f +_c g) \]
\[ e +_b f = b; e +_b f \]

Composition

\[ b; c = b \text{ and } c \]
\[ 0; e = e; 0 = 0 \]
\[ 1; e = e; 1 = e \]
\[ e; (f; g) = (e; f); g \]
\[ (e +_b f); g = e; g +_b f; g \]

fizzbuzz1

\[ = p; (((q; u + \text{not } d \ r; u) +_c (s; u +_d t; u))^{(b)}; v \]
\[ = p; (((q + \text{not } d \ r) +_c (s +_d t)); u)^{(b)}; v \]
\[ = p; (((r +_d q) +_c (s +_d t)); u)^{(b)}; v \]
\[ = \text{fizzbuzz2} \]

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

**Conditionals**

\[ e +_b e = e \]
\[ e +_b f = f + \text{not } b \ e \]
\[ (e +_b f) +_c g = e +_b \text{and } c \ (f +_c g) \]
\[ e +_b f = b; e +_b f \]

**Composition**

\[ b; c = b \text{ and } c \]
\[ 0; e = e; 0 = 0 \]
\[ 1; e = e; 1 = e \]
\[ e; (f; g) = (e; f); g \]
\[ (e +_b f); g = e; g +_b f; g \]

**fizzbuzz1**

\[ = p; (((q; u + \text{not } d \ r; u) +_c (s; u + d \ t; u))^{(b)}; v \]
\[ = p; (((q + \text{not } d \ r) +_c (s + d \ t)); u)^{(b)}; v \]
\[ = p; (((r + d \ q) +_c (s + d \ t)); u)^{(b)}; v \]
\[ = p; (((r + \text{and } c \ q +_c (s + d \ t))); u)^{(b)}; v \]
\[ = \text{fizzbuzz2} \]

(\text{Smolka, Foster, Hsu, K., Kozen & Silva 2019})
Axiomatizing our reasoning

**Conditionals**

\[ e +_b e = e \]
\[ e +_b f = f + \text{not } b \ e \]
\[ (e +_b f) +_c g = e +_b \text{ and } c \ (f +_c g) \]
\[ e +_b f = b; e +_b f \]

**Composition**

\[ b; c = b \text{ and } c \]
\[ 0; e = e; 0 = 0 \]
\[ 1; e = e; 1 = e \]
\[ e; (f; g) = (e; f); g \]
\[ (e +_b f); g = e; g +_b f; g \]

**Loops**

\[ e^{(b)}; f = e; (e^{(b)}; f) +_b f \]

\[ g = e; g +_b f \quad \text{e productive} \]

\[ g = e^{(b)}; f \]

\[ \ldots \text{and generalizations of the above} \]

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

**Conditionals**

\[
e +_b e = e
\]

\[
e +_b f = f + \text{not } b \ e
\]

\[
(e +_b f) +_c g = e +_b \ e \text{ and } c \ (f +_c g)
\]

\[
e +_b f = b; e +_b f
\]

**Composition**

\[
b; c = b \text{ and } c
\]

\[
0; e = e; 0 = 0
\]

\[
1; e = e; 1 = e
\]

\[
e; (f; g) = (e; f); g
\]

\[
(e +_b f); g = e; g +_b f; g
\]

**Loops**

\[
e^{(b)}; f = e; (e^{(b)}; f) +_b f
\]

\[
g = e; g +_b f \quad \text{e productive}
\]

\[
g = e^{(b)}; f
\]

\[
\ldots \text{ and generalizations of the above}
\]

---

**Open Question #1**

Do we need the generalized loop rules?

---

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Axiomatizing our reasoning

**Conditionals**

$$e +_b e = e$$
$$e +_b f = f + \text{not } b \ e$$
$$(e +_b f) +_c g = e +_b \text{ and } c \ (f +_c g)$$
$$e +_b f = b; e +_b f$$

**Composition**

$$b; c = b \text{ and } c$$
$$0; e = e; 0 = 0$$
$$1; e = e; 1 = e$$
$$e; (f; g) = (e; f); g$$
$$(e +_b f); g = e; g +_b f; g$$

**Loops**

$$e^{(b)}; f = e; (e^{(b)}; f) +_b f$$

$$g = e; g +_b f \quad \text{e productive}$$

$$g = e^{(b)}; f$$

... and generalizations of the above

---

**Open Question #1**

Do we need the generalized loop rules?

**Open Question #2**

Can we factor out the side condition?

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)
Why is GKAT so hard?

Complete algebraic axiomatization of KAT goes back to Kozen (1996)…
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GKAT programs are a proper subset of KAT programs...
Why is GKAT so hard?

Complete algebraic axiomatization of KAT goes back to Kozen (1996) . . .

GKAT programs are a proper subset of KAT programs . . .

So why is GKAT so difficult?
Kleene Algebra with Tests

Not every deterministic KAT program is a GKAT program.
Kleene Algebra with Tests

Not every deterministic KAT program is a GKAT program.

Example (Schmid, K., Kozen & Silva 2021), (Kozen & Tseng 2008)

```
while b do
    p;
    if b then break;
    p
```
A similar problem in process algebra

Milner studied *regular expressions up to bisimilarity* in 1984.
A similar problem in process algebra

Milner studied *regular expressions up to bisimilarity* in 1984.

He proposed axioms for equivalence, but left completeness open.
A similar problem in process algebra

Not all behaviors realized as expressions (Milner 1984), (Bosscher 1997)

\[
\mu x(b + a \cdot (c + a \cdot x))
\]
A similar problem in process algebra

Not all behaviors realized as expressions (Milner 1984), (Bosscher 1997)

\[ \mu x (b + a \cdot (c + a \cdot x)) \]

Axiomatization for fragment without 1 (Grabmayer & Fokkink 2020)

\[ 0 \mid a \mid e + f \mid e; f \mid e^* f \]
Introducing... Skip-free GKAT!

The *skip-free fragment* of GKAT is given by

\[
0 \mid p \mid e +_b f \mid e; f \mid e^{(b)}f
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- Can still express a wide range of programs...

\[
\text{fizzbuzz1} = p; ((r; u + \text{not} d q; u) + c (s; u + d t; u))^{(b)} v
\]

\[
\text{fizzbuzz2} = p; ((q + d \text{and} c (r + c (s + d t)); u))^{(b)} v
\]
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\[ 0 \mid p \mid e + b \ f \mid e ; f \mid e^{(b)} f \]

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\[
\text{fizzbuzz2} = p; ((q + d \text{ and} c (r + c (s + d t)))u)^{(b)} v
\]

- Every skip-free expression satisfies the side condition!

\[
\frac{g = e; g + b f \quad \text{e productive}}{g = e^{(b)} ; f} \implies \frac{g = e; g + b f}{g = e^{(b)} f}
\]
Axioms for Skip-free GKAT

**Conditionals**

\[ e + b \ e = e \]

\[ e + b \ f = f + \text{not } b \ e \]

\[ (e + b \ f) + c \ g = e + b \ \text{and } c \ (f + c \ g) \]

**Composition**

\[ 0; e = e; 0 = 0 \]

\[ e; (f; g) = (e; f); g \]

\[ (e + b \ f); g = e; g + b \ f; g \]

**Loops**

\[ e^{(b)} f = e; (e^{(b)} f) + b \ f \]

\[ g = e; g + b \ f \]

\[ g = e^{(b)} f \]
Axioms for Skip-free GKAT

Conditionals

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\[ (e +_b f) +_c g = e +_b \text{ and } c \ (f +_c g) \]

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\[ e^{(b)} f = e; (e^{(b)} f) +_b f \]
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\[ g = e^{(b)} f \]

Completeness Theorem
(K., Schmid & Silva 2023)

For \( e, f \) skip-free GKAT expressions, the following are equivalent:

1. \( e \) and \( f \) are language equivalent
2. the equation \( e = f \) is provable
Axioms for Skip-free GKAT

**Conditionals**

\[ e +_b e = e \]
\[ e +_b f = f + \text{not } b \ e \]
\[ (e +_b f) +_c g = e +_b \text{ and } c \ (f +_c g) \]

**Composition**

\[ 0; e = e; 0 = 0 \]
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**Loops**

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This is an *algebraic and finitary* axiomatization!
Axioms for Skip-free GKAT

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For $e, f$ skip-free GKAT expressions, the following are equivalent:

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Proof sketch.

We designed two transformations:

$gtr$: 1-free GKAT expressions $\to$ fragment of 1-free regex

$rtg$: fragment of 1-free regex $\to$ 1-free GKAT expressions

Given 1-free GKAT expressions $e$ and $f$ with $J_e K$ $J_f K$:

$J_{gtr} pe q K$ $J_{gtr} pf q K$ $\equiv_{gtr} p e q$ $\equiv_{gtr} p f q$

$rtg p gtr p e q q$ $rtg p gtr p f q q$ $\equiv_{rtg} e f$
Axioms for Skip-free GKAT

**Completeness Theorem** *(K., Schmid & Silva 2023)*

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Given 1-free GKAT expressions $e$ and $f$ with $J_e K_e \subseteq J_f K_f$:

$$\begin{align*}
J_{gtr} p e q K_{gtr} p f q \quad \text{if} \quad gtr p e q \\
rtg p gtr p e q \quad \text{if} \quad rtg p gtr p f q \\
e = f
\end{align*}$$
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Given 1-free GKAT expressions $e$ and $f$ with $\llbracket e \rrbracket = \llbracket f \rrbracket$:

$$\llbracket gtr(e) \rrbracket = \llbracket gtr(f) \rrbracket$$
Axioms for Skip-free GKAT

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Given 1-free GKAT expressions $e$ and $f$ with $[e] = [f]$:

$$[\text{gtr}(e)] = [\text{gtr}(f)] \implies \text{gtr}(e) \equiv \text{gtr}(f)$$
Axioms for Skip-free GKAT

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$$[gtr(e)] = [gtr(f)] \implies gtr(e) \equiv gtr(f) \implies rtg(gtr(e)) \equiv rtg(gtr(f)) \implies e \equiv f$$
Future work

Regex/bisimilarity
(Milner 1984)
Future work

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Completeness of 1-free regex/bisimilarity
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While fragment of KAT
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← reduction →
Future work

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GKAT
(Smolka et al, 2019)

Completeness of skip-free GKAT

Completeness of GKAT?
Recap

- GKAT: Propositional while programs/language equivalence (Smolka et al. 2019)
- **Open problem:** Is finite axiomatization of GKAT complete?
- Similar problem in process algebra (Milner 1984) — open for 38 years!
- Inspired by (Grabmayer & Fokkink 2020) we introduce skip-free GKAT

\[ 0 | a | e + f | e; f | e^* f \quad \implies \quad 0 | p | e +_b f | e; f \mid e^{(b)} f \]

- **Theorem:** Finite axiomatization is complete for skip-free GKAT
  - Completeness proof is a reduction to (Grabmayer & Fokkink 2020)
- **New question:** Can we reduce all of GKAT to regex/bisimilarity?

Questions are welcome!