A Complete Inference System for Skip-free Guarded Kleene Algebra with Tests

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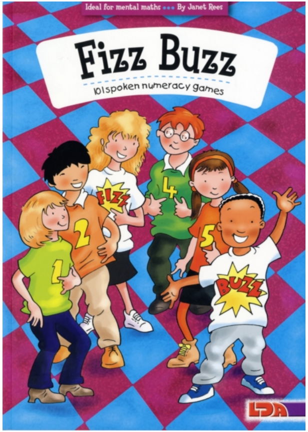
* who kindly let me use his slides

Reasoning about software: A story

Let's play a game of *Fizzbuzz!*

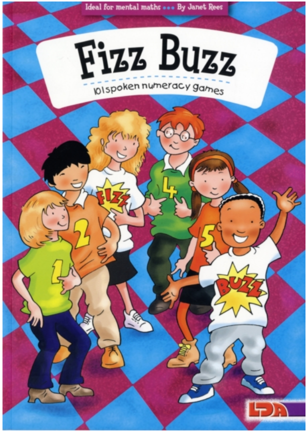
- Take turns counting to 100.
- Number divisible by 3 $\Rightarrow$ say *Fizz!*
- Number divisible by 5 $\Rightarrow$ say *Buzz!*
- Number is divisible by 3 *and* 5 $\Rightarrow$ say *Fizzbuzz!*
- Otherwise, just say the number.



Reasoning about software: A story

Let's play a game of *Fizzbuzz!*

- Take turns counting to 100.
- Number divisible by 3 (but not 5) $\Rightarrow$ say *Fizz!*
- Number divisible by 5 (but not 3) $\Rightarrow$ say *Buzz!*
- Number is divisible by 3 *and* 5 $\Rightarrow$ say *Fizzbuzz!*
- Otherwise, just say the number.



Reasoning about software: A story

```
def fizzbuzz1 =
  n := 1;
  while n ⩽ 100 do
    if 3|n then
      if not 5|n then
        print fizz; n++;
      else
        print fizzbuzz; n++;
    else if 5|n then
      print buzz; n++;
    else
      print n; n++;
  print done!;
```

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      print buzz; n++;
    else
      print n; n++;
  print done!;
```

```
def fizzbuzz2 =
  n := 1;
  while n ⩽ 100 do
    if 5|n and 3|n then
      print fizzbuzz;
    else if 3|n then
      print fizz;
    else if 5|n then
      print buzz;
    else
      print n;
    n++;
  print done!;
```

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```

$$\text{fizzbuzz1} \stackrel{?}{=} \text{fizzbuzz2}$$

Reasoning about software: A story

Starting with fizzbuzz1...

```
n := 1;
while n ⩽ 100 do
   if 3|n then
      if not 5|n then
         print fizz; n++;
      else
         print fizzbuzz; n++;
   else if 5|n then
      print buzz; n++;
   else
      print n; n++;
print done!;
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Reasoning about software: A story

Move the n++; to the end...

```
n := 1;
while n ⩽ 100 do
   if 3|n then
      if not 5|n then
         print fizz;
      else
         print fizzbuzz;
   else if 5|n then
      print buzz;
   else
      print n;
    n++;
print done!;
```

Reasoning about software: A story

Negate not $5|n$ and flip the branches

```
n := 1;
while n ⩽ 100 do
    if 3|n then
        if 5|n then
            print fizzbuzz;
        else
            print fizz;
    else if 5|n then
        print buzz;
    else
        print n;
    n++;
print done!;
```

Reasoning about software: A story

Merge $3|n$ and $5|n$

```
n := 1;
while n ≤ 100 do
   if 3|n and 5|n then
      print fizzbuzz;
   else if 3|n then
      print fizz;
   else if 5|n then
      print buzz;
   else
      print n;
    n++;
print done!;
```

Reasoning about software: A story

This is precisely fizzbuzz2!

```
n := 1;
while n ⩽ 100 do
   if 3|n and 5|n then
      print fizzbuzz;
   else if 3|n then
      print fizz;
   else if 5|n then
      print buzz;
   else
      print n;
    n++;
print done!;
```

Taking a step back

- The reasoning steps applied are very general. For instance:

$$\boxed{\text{if not b then e else f}} = \boxed{\text{if b then f else e}}$$

should work regardless of what b, e and f are.

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  $$\boxed{\text{if not b then e else f}} = \boxed{\text{if b then f else e}}$$

  should work regardless of what b, e and f are.

- We treated the program as an expression, and reasoned equationally.

  > *programs are mathematical expressions, [...] subject to a set of laws as rich and elegant as those of any other branch of mathematics* (Hoare et al. 1984)

Taking a step back

By abstracting away from individual actions and tests, we go from . . .

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      print fizz;
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      print buzz;
    else
      print n;
    n++;
  print done!;
```

Taking a step back

... to *propositional* programs:

```
def fizzbuzz1 =
  p;
  while b do
    if c then
      if not d then
        r; u;
      else
        q; u;
    else if d then
      s; u;
    else
      t; u;
  v;
```

```
def fizzbuzz2 =
  p;
  while b do
    if d and c then
      q;
    else if c then
      r;
    else if d then
      s;
    else
      t;
    u;
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```

Formalizing our reasoning

$$\boxed{\text{if b then e else f}} = \boxed{\text{if not b then f else e}} \qquad (\textit{skew commutativity})$$

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$$\boxed{\text{if b then e else f}} \quad = \quad \boxed{\text{if not b then f else e}} \qquad (\textit{skew commutativity})$$

$$\boxed{\text{(if b then e else f);g}} \quad = \quad \boxed{\text{if b then e;g else f;g}} \qquad (\textit{left distributivity})$$

Formalizing our reasoning

if b then e else f	=	if not b then f else e	(*skew commutativity*)
(if b then e else f);g	=	if b then e;g else f;g	(*left distributivity*)
if b then (if c then e else f) else g	=	if b and c then e else (if b then f else g)	(*skew associativity*)

```
if b then
    (if c then e else f)
else
    g
```

=

```
if b and c then
    e
else
    (if b then f else g)
```

Formalizing our reasoning

if b then e else f	=	if not b then f else e	(*skew commutativity*)
(if b then e else f);g	=	if b then e;g else f;g	(*left distributivity*)

if b then
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	=	if b and c then
		e
		else
		(if b then f else g)
			(*skew associativity*)

while b do
e
	=	if b then
		e
		while b do
		e
			(*loop unrolling*)

Formalizing our reasoning: Kleene Algebra with Tests

Fix a set $\{p, q, \ldots\}$ of actions and a Boolean algebra $\{b, c, \ldots\}$ of tests

$$b \mid p \mid e + f \mid e; f \mid e^*$$

- Extends regular expressions with a Boolean algebra of tests

$$\boxed{\text{if } b \text{ then e else f}} = b; e + (\text{not } b); f$$

$$\boxed{\text{while } b \text{ do e}} = (b; e)^* ; (\text{not } b)$$

(Kozen 1996), (Kozen & Smith 1996)

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- Language semantics in terms of *guarded strings*: $\alpha_1 p_1 \alpha_2 p_2 \alpha_3 p_3 \cdots \alpha_n p_n \alpha_{n+1}$

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- Complete and finitary axiomatization
- Non-determinism makes equivalence PSPACE-complete

(Kozen 1996), (Kozen & Smith 1996)

Formalizing our reasoning: Guarded KAT

Fix a set $\{\mathsf{p}, \mathsf{q}, \dots\}$ of actions and a Boolean algebra $\{\mathsf{b}, \mathsf{c}, \dots\}$ of tests

$$\mathsf{b} \mid \mathsf{p} \mid e +_{\mathsf{b}} f \mid e; f \mid e^{(\mathsf{b})}$$

The part of KAT specifically for while programs!

(Kozen & Tseng 2008), (Smolka, Foster, Hsu, K., Kozen & Silva 2019)

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$$e +_{\mathsf{b}} f = \text{if } \mathsf{b} \text{ then e else f} \qquad e^{(\mathsf{b})} = \text{while } \mathsf{b} \text{ do e}$$

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$$\begin{aligned} e +_b f &= \text{if } b \text{ then e else f} \\ &= b; e + (\text{not } b); f \end{aligned} \qquad \begin{aligned} e^{(b)} &= \text{while } b \text{ do e} \\ &= (b; e)^* ; (\text{not } b) \end{aligned}$$

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$$\boxed{\text{fizzbuzz1}} = \mathsf{p}; ((\mathsf{r}; \mathsf{u} +_{\mathsf{not\ d}} \mathsf{q}; \mathsf{u}) +_{\mathsf{c}} (\mathsf{s}; \mathsf{u} +_{\mathsf{d}} \mathsf{t}; \mathsf{u}))^{(\mathsf{b})}; \mathsf{v}$$

$$\boxed{\text{fizzbuzz2}} = \mathsf{p}; ((\mathsf{q} +_{\mathsf{d\ and\ c}} (\mathsf{r} +_{\mathsf{c}} (\mathsf{s} +_{\mathsf{d}} \mathsf{t})))\mathsf{u})^{(\mathsf{b})}; \mathsf{v}$$

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- Complete but infinitary axiomatization

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Axiomatizing our reasoning

Conditionals

$$e +_{\mathsf{b}} e = e$$

$$e +_{\mathsf{b}} f = f +_{\mathsf{not\ b}} e$$

$$(e +_{\mathsf{b}} f) +_{\mathsf{c}} g = e +_{\mathsf{b\ and\ c}} (f +_{\mathsf{c}} g)$$

$$e +_{\mathsf{b}} f = \mathsf{b}; e +_{\mathsf{b}} f$$

Composition

$$\mathsf{b}; \mathsf{c} = \mathsf{b\ and\ c}$$

$$\mathsf{0}; e = e; \mathsf{0} = \mathsf{0}$$

$$\mathsf{1}; e = e; \mathsf{1} = e$$

$$e; (f; g) = (e; f); g$$

$$(e +_{\mathsf{b}} f); g = e; g +_{\mathsf{b}} f; g$$

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)

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fizzbuzz1

$$= \mathsf{p}; ((\mathsf{q}; \mathsf{u} +_{\mathsf{not\ d}} \mathsf{r}; \mathsf{u}) +_{\mathsf{c}} (\mathsf{s}; \mathsf{u} +_{\mathsf{d}} \mathsf{t}; \mathsf{u}))^{(\mathsf{b})}; \mathsf{v}$$

$= \mathsf{fizzbuzz2}$

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)

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Loops

$$e^{(\mathsf{b})}; f = e; (e^{(\mathsf{b})}; f) +_{\mathsf{b}} f$$

$$\frac{g = e; g +_{\mathsf{b}} f \quad e \text{ productive}}{g = e^{(\mathsf{b})}; f}$$

. . . and generalizations of the above

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Open Question #1

Do we need the generalized loop rules?

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)

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...and generalizations of the above

Open Question #1

Do we need the generalized loop rules?

Open Question #2

Can we factor out the side condition?

(Smolka, Foster, Hsu, K., Kozen & Silva 2019)

Why is GKAT so hard?

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GKAT programs are a proper subset of KAT programs. . .

So why is GKAT so dificult?

Kleene Algebra with Tests

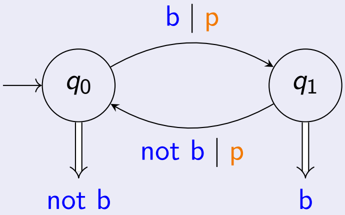
Not every deterministic KAT program is a GKAT program.

Kleene Algebra with Tests

Not every deterministic KAT program is a GKAT program.

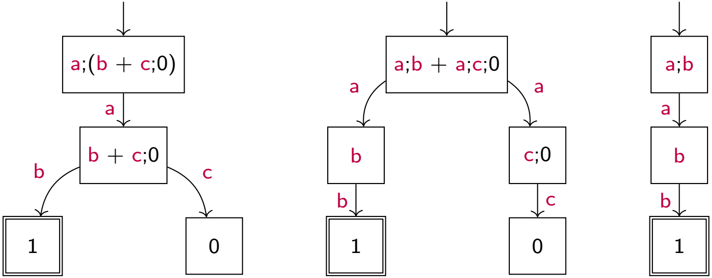
Example (Schmid, K., Kozen & Silva 2021), (Kozen & Tseng 2008)

```
while b do
    p;
    if b then break;
    p
```



A similar problem in process algebra

Milner studied *regular expressions up to bisimilarity* in 1984.



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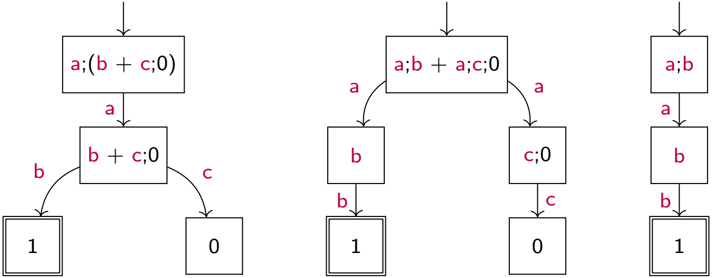
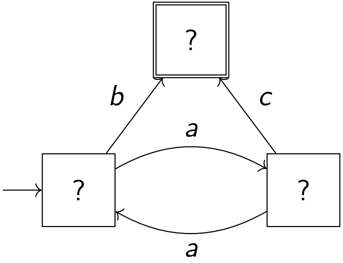


He proposed axioms for equivalence, but left completeness open.

A similar problem in process algebra

Not all behaviors realized as expressions (Milner 1984), (Bosscher 1997)



$$\mu x(b + a \cdot (c + a \cdot x))$$

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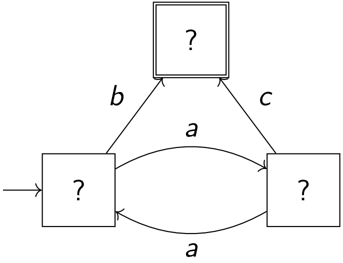

$$\mu x(b + a \cdot (c + a \cdot x))$$

Axiomatization for fragment *without 1* (Grabmayer & Fokkink 2020)

$$0 \mid a \mid e + f \mid e; f \mid e^* f$$

Introducing...Skip-free GKAT!

The *skip-free fragment* of GKAT is given by

$$0 \mid \mathsf{p} \mid e +_{\mathsf{b}} f \mid e \mathbin{;} f \mid e^{(\mathsf{b})} f$$

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$$\boxed{\text{fizzbuzz1}} = \mathsf{p}; ((\mathsf{r}; \mathsf{u} +_{\mathsf{not\ d}} \mathsf{q}; \mathsf{u}) +_{\mathsf{c}} (\mathsf{s}; \mathsf{u} +_{\mathsf{d}} \mathsf{t}; \mathsf{u}))^{(\mathsf{b})} \mathsf{v}$$

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- Every skip-free expression satisfies the side condition!

$$\frac{g = e; g +_{\mathsf{b}} f \quad e \text{ productive}}{g = e^{(\mathsf{b})}; f} \implies \frac{g = e; g +_{\mathsf{b}} f}{g = e^{(\mathsf{b})} f}$$

Axioms for Skip-free GKAT

Loops

$$e^{(b)}f = e; (e^{(b)}f) +_b f$$

$$\frac{g = e; g +_b f}{g = e^{(b)}f}$$

Conditionals

$$e +_b e = e$$

$$e +_b f = f +_{\text{not } b} e$$

$$(e +_b f) +_c g = e +_{b \text{ and } c} (f +_c g)$$

Composition

$$0; e = e; 0 = 0$$

$$e; (f; g) = (e; f); g$$

$$(e +_b f); g = e; g +_b f; g$$

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$$\frac{g = e; g +_{\mathsf{b}} f}{g = e^{(\mathsf{b})} f}$$

Completeness Theorem

(K., Schmid & Silva 2023)

For e, f skip-free GKAT expressions, the following are equivalent:

1. e and f are language equivalent
2. the equation $e = f$ is provable

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This is an *algebraic* and *finitary* axiomatization!

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Proof sketch.

We designed two transformations:

- *gtr*: 1-free GKAT expressions $\rightarrow$ fragment of 1-free regex
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Given 1-free GKAT expressions e and f with $[\![e]\!] = [\![f]\!]$:

$$[\![gtr(e)]\!] = [\![gtr(f)]\!]$$

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Future work

Regex/bisimilarity

(Milner 1984)

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⋮

Completeness of
1-free regex/bisimilarity

(Grabmayer & Fokkink 2020)

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**Completeness of
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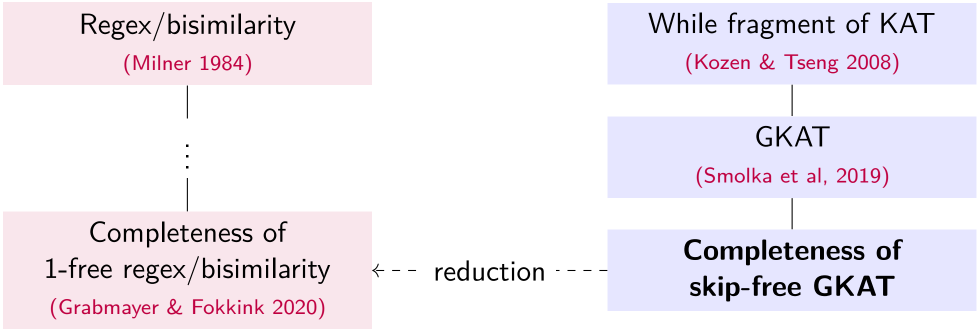
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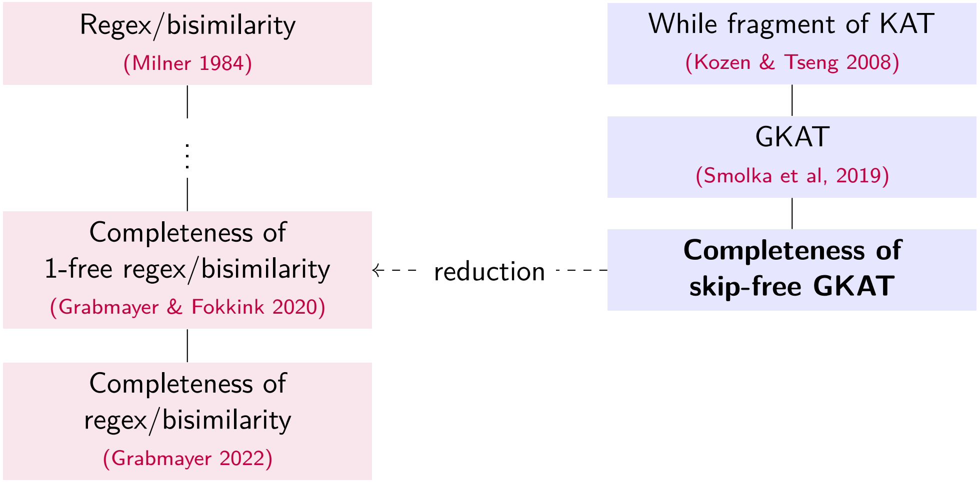

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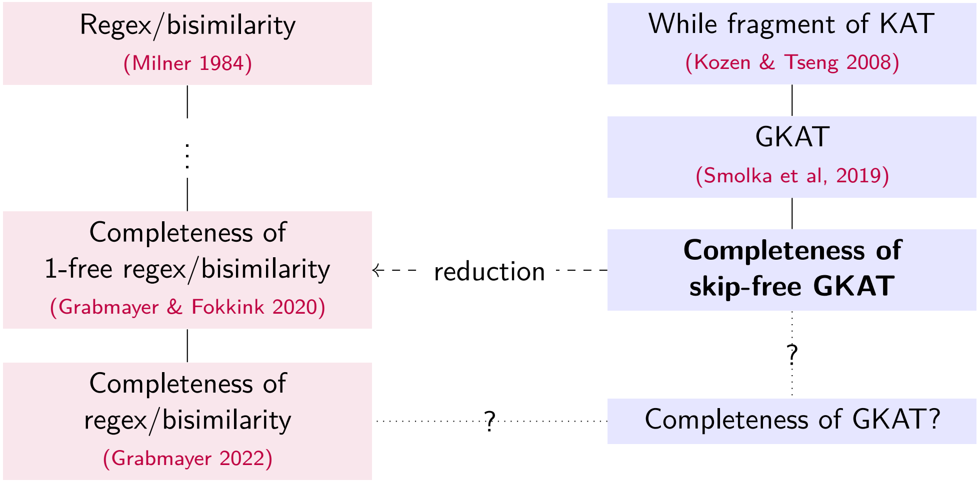

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Completeness of GKAT?

Recap

- GKAT: Propositional while programs/language equivalence (Smolka et al. 2019)
- **Open problem:** Is finite axiomatization of GKAT complete?
- Similar problem in process algebra (Milner 1984) — open for 38 years!
- Inspired by (Grabmayer & Fokkink 2020) we introduce skip-free GKAT

$$0 \mid a \mid e + f \mid e ; f \mid e^* f \qquad \Longrightarrow \qquad 0 \mid \mathsf{p} \mid e +_{\mathsf{b}} f \mid e ; f \mid e^{(\mathsf{b})} f$$

- **Theorem:** Finite axiomatization is complete for skip-free GKAT
  - Completeness proof is a reduction to (Grabmayer & Fokkink 2020)
- **New question:** Can we reduce all of GKAT to regex/bisimilarity?

Questions are welcome!