Kleene Algebra — Lecture 5

ESSLLI 2023

Last lecture

- We built a method to convert automata to expressions.

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- Along the way, we developed some linear algebra over expressions.

Today's lecture

Finally: the completeness proof.

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1. Bisimilar states yield provably equivalent expressions.

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2. Going from expressions to automata and back preserves equivalence.

Today's lecture

Finally: the completeness proof.

Three crucial insights:

1. Bisimilar states yield provably equivalent expressions.
2. Going from expressions to automata and back preserves equivalence.
3. Relation between solutions of an automaton and its powerset automaton.

Matrices and bisimulations

Definition

Let S_1 and S_2. An S_1*-by-*S_2 *matrix* is a function $M : S_1 \times S_2 \to \mathbb{E}$.

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Let M be an S_1-by-S_2 matrix, and N an S_2-by-S_3 matrix.

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Let M be an S_1-by-S_2 matrix, and N an S_2-by-S_3 matrix.
Now $M \cdot N$ is the S_1-by-S_3 matrix where

$$(M \cdot N)(s_1, s_3) = \sum_{s_2 \in S_2} M(s_1, s_2) \cdot N(s_2, s_3)$$

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If b is an S_2-vector, then $M \cdot b$ is the S_1-vector given by

$$(M \cdot b)(s_1) = \sum_{s_2 \in S_2} M(s_1, s_2) \cdot b(s_2)$$

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$$(M \cdot b)(s_1) = \sum_{s_2 \in S_2} M(s_1, s_2) \cdot b(s_2)$$

We write M^T for the *transpose* of M, i.e., the S_2-by-S_1 matrix where

$$M^T(s_2, s_1) = M(s_1, s_2)$$

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Definition

Let S_1 and S_2 be sets, and let $R \subseteq S_1 \times S_2$ be a relation between S_1 and S_2.

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Lemma

Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ be an automaton for $i \in \{0, 1\}$.

Furthermore, let $R \subseteq Q_0 \times Q_1$ be a simulation of A_0 by A_1.

Then $M_R^T \cdot b_{A_0} \leqq b_{A_1}$.

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Proof.

This comes down to showing that

$$\forall q_1 \in Q_1. \sum_{q_0 \in Q_0} M_R^T(q_1, q_0) \cdot b_{A_0}(q_0) \leqq b_{A_1}(q_1)$$

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Proof.

The claim works out to be equivalent to

$$\forall q_1 \in Q_1, q_0 \in Q_0. \sum_{q_0' \in Q_0} M_R^T(q_1, q_0') \cdot M_{A_0}(q_0', q_0) \leqq \sum_{q_1' \in Q_1} M_{A_1}(q_1, q_1') \cdot M_R^T(q_1', q_0)$$

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Suppose $q_1 \in Q_1$, $q_0, q_0' \in Q_0$ such that $q_0' \xrightarrow{\mathtt{a}} q_0$ and $q_0' \; R \; q_1$.

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Then there exists q_1' such that $q_1 \xrightarrow{\mathtt{a}} q_1'$ and $q_1 \; R \; q_1'$. Thus $\mathtt{a} \cdot [q_0 \; R \; q_1']$ is a term in the sum on the right.

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Proof.

The claim works out to be equivalent to

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Suppose $q_1 \in Q_1$, $q_0, q_0' \in Q_0$ such that $q_0' \xrightarrow{\mathtt{a}} q_0$ and $q_0' \; R \; q_1$.

Then there exists q_1' such that $q_1 \xrightarrow{\mathtt{a}} q_1'$ and $q_1 \; R \; q_1'$. Thus $\mathtt{a} \cdot [q_0 \; R \; q_1']$ is a term in the sum on the right. Since $[q_0 \; R \; q_1'] \cdot \mathtt{a} \equiv \mathtt{a} \cdot [q_0' \; R \; q_1]$, we are done. □

Matrices and bisimulations

Corollary

Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ *be an automaton for* $i \in \{0, 1\}$.

Furthermore, let R *be a simulation of* A_0 *by* A_1.

Then $M_R^T \cdot M_{A_0}^* \leqq M_{A_1}^* \cdot M_R^T$.

Proof sketch.

In KA, if $e \cdot f \leqq g \cdot e$, then $e \cdot f^* \leqq g^* \cdot e$ (see exercises).

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In KA, if $e \cdot f \leqq g \cdot e$, then $e \cdot f^* \leqq g^* \cdot e$ (see exercises).

Recall that and $M_R^T \cdot M_{A_0} \leqq M_{A_1} \cdot M_R^T$ by the last lemma.

Matrices and bisimulations

Corollary

Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ *be an automaton for* $i \in \{0, 1\}$*.*

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In KA, if $e \cdot f \leqq g \cdot e$, then $e \cdot f^* \leqq g^* \cdot e$ (see exercises).

Recall that and $M_R^T \cdot M_{A_0} \leqq M_{A_1} \cdot M_R^T$ by the last lemma.

Matrices over KA follow the laws of KA, so the claim follows. □

Matrices and bisimulations

Theorem

Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ *be an automaton for* $i \in \{0, 1\}$.

Also, let R *be a simulation of* A_0 *by* A_1.

Finally, let $e_0 = (M_{A_0}^* \cdot b_{A_0})(q_0)$ *and* $e_1 = (M_{A_1}^* \cdot b_{A_1})(q_1)$.

If $q_0 \mathrel{R} q_1$, *then* $e_0 \leqq e_1$.

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If $q_0 \mathrel{R} q_1$, *then* $e_0 \leqq e_1$.

Proof.

Derive as follows:

$$e_0 = (M^*_{A_0} \cdot b_{A_0})(q_0)$$

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If $q_0 \; R \; q_1$*, then* $e_0 \leqq e_1$*.*

Proof.

Derive as follows:

$$e_0 \equiv [q_0 \; R \; q_1] \cdot (M^*_{A_0} \cdot b_{A_0})(q_0)$$

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Proof.

Derive as follows:

$$e_0 \leqq \sum_{q_0' \in Q_0} [q_0' \mathrel{R} q_1] \cdot (M^*_{A_0} \cdot b_{A_0})(q_0')$$

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Proof.

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$$e_0 \leqq \sum_{q'_0 \in Q_0} M_R^T(q_1, q'_0) \cdot (M^*_{A_0} \cdot b_{A_0})(q'_0)$$

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If $q_0 \; R \; q_1$, then $e_0 \leqq e_1$.

Proof.

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Proof.

Derive as follows:

$$e_0 \leqq (M^*_{A_1} \cdot b_{A_1})(q_1) = e_1$$

□

The round-trip theorem — left-to-right

Let $e \in \mathbb{E}$. We write $K(e)$ for $(M^*_{A_e} \cdot b_e)(e)$.

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Proof.

Note that $M^*_{A_e} \cdot b_e$, as an $\hat{\rho}(e)$-vector, is the least solution to A_e.

So, if we have a solution s to A_e, then $M^*_{A_e} \cdot b_e \leqq s$.

We now choose $\hat{\rho}(e)$-vector s by setting $s(e) = e$.

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We now choose $\hat{\rho}(e)$-vector s by setting $s(e) = e$.

Now s is a solution to A_e, by the fundamental theorem:

$$[e \in \mathbb{A}] + \sum_{e \xrightarrow{\mathtt{a}}_{\mathbb{E}} e'} \mathtt{a} \cdot e' \leqq e$$

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□

The round-trip theorem — right-to-left

Lemma

The following hold for all $e, f, g \in \mathbb{E}$:

$$K(e), K(f) \leqq K(e + f) \qquad K(e \cdot g + f \cdot g) \leqq K((e + f) \cdot g)$$

$$K(e \cdot (f \cdot g)) \leqq K((e \cdot f) \cdot g) \qquad K((1 + e \cdot e^*) \cdot f) \leqq K(e^* \cdot f)$$

$$K(1 \cdot e) \leqq K(e) \qquad K(e \cdot 1) \leqq K(e)$$

The round-trip theorem — right-to-left

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The following hold for all $e, f, g \in \mathbb{E}$:

$$K(e), K(f) \leqq K(e+f) \qquad K(e \cdot g + f \cdot g) \leqq K((e+f) \cdot g)$$

$$K(e \cdot (f \cdot g)) \leqq K((e \cdot f) \cdot g) \qquad K((1 + e \cdot e^*) \cdot f) \leqq K(e^* \cdot f)$$

$$K(1 \cdot e) \leqq K(e) \qquad K(e \cdot 1) \leqq K(e)$$

Proof sketch.

By the result about similarity and solutions!

The round-trip theorem — right-to-left

Lemma

The following hold for all $e, f, g \in \mathbb{E}$:

$$K(e), K(f) \leqq K(e+f) \qquad K(e \cdot g + f \cdot g) \leqq K((e+f) \cdot g)$$

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Proof sketch.

By the result about similarity and solutions!

For instance, e in A_e is simulated by $e + f$ in A_{e+f}. □

The round-trip theorem — right-to-left

Lemma

Let $\mathtt{e} \in \mathbb{E}$ *and* $\mathtt{a} \in \Sigma$. *Now* $\mathtt{a} \cdot K(e) \leqq K(\mathtt{a} \cdot e)$.

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$ *and* $\mathtt{a} \in \Sigma$. *Now* $\mathtt{a} \cdot K(e) \leqq K(\mathtt{a} \cdot e)$.

Proof.

First, note that $1 \cdot e$ (in $A_{\mathtt{a} \cdot e}$) simulates e (in A_e); so

$$K(e) = (M_{A_e}^{*} \cdot b_e)(e) \leqq (M_{A_{\mathtt{a} \cdot e}}^{*} \cdot b_{\mathtt{a} \cdot e})(1 \cdot e)$$

The round-trip theorem — right-to-left

Lemma

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We derive:

$$\mathtt{a} \cdot K(e) \leqq \mathtt{a} \cdot (M_{A_{\mathtt{a} \cdot e}}^* \cdot b_{\mathtt{a} \cdot e})(1 \cdot e)$$

The round-trip theorem — right-to-left

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We derive:

$$\begin{aligned} \mathtt{a} \cdot K(e) &\leqq \mathtt{a} \cdot (M_{A_{\mathtt{a} \cdot e}}^* \cdot b_{\mathtt{a} \cdot e})(1 \cdot e) \\ &\leqq M_{A_{\mathtt{a} \cdot e}}(\mathtt{a} \cdot e, 1 \cdot e) \cdot (M_{A_{\mathtt{a} \cdot e}}^* \cdot b_{A_{\mathtt{a} \cdot e}})(1 \cdot e) \end{aligned}$$

The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$ *and* $\mathtt{a} \in \Sigma$. *Now* $\mathtt{a} \cdot K(e) \leqq K(\mathtt{a} \cdot e)$.

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We derive:

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The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$ *and* $\mathtt{a} \in \Sigma$. *Now* $\mathtt{a} \cdot K(e) \leqq K(\mathtt{a} \cdot e)$.

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We derive:

$$\begin{aligned}
\mathtt{a} \cdot K(e) &\leqq \mathtt{a} \cdot (M^*_{A_{\mathtt{a} \cdot e}} \cdot b_{\mathtt{a} \cdot e})(1 \cdot e) \\
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&\leqq (M_{A_{\mathtt{a} \cdot e}} \cdot M^*_{A_{\mathtt{a} \cdot e}} \cdot b_{A_{\mathtt{a} \cdot e}})(\mathtt{a} \cdot e) \\
&\leqq (M^*_{A_{\mathtt{a} \cdot e}} \cdot b_{A_{\mathtt{a} \cdot e}})(\mathtt{a} \cdot e) = K(\mathtt{a} \cdot e)
\end{aligned}$$

□

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$*. Now* $e \leqq K(e)$*.*

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. *Now* $e \leqq K(e)$.

Proof.

We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$.

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

Proof.

We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Induction on e; in the base:

- If $e = 0$, then $e \cdot K(f) \equiv 0$, so the claim holds immediately.

The round-trip theorem — right-to-left

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- If $e = 0$, then $e \cdot K(f) \equiv 0$, so the claim holds immediately.
- If $e = 1$, then we derive:

$$e \cdot K(f) = 1 \cdot K(f) \equiv K(f) \leqq K(1 \cdot f) = K(e \cdot f)$$

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. *Now* $e \leqq K(e)$.

Proof.

We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Induction on e; in the base:

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- If $e = 1$, then we derive:

$$e \cdot K(f) = 1 \cdot K(f) \equiv K(f) \leqq K(1 \cdot f) = K(e \cdot f)$$

- If $e = \mathtt{a}$, then by the previous lemma we derive:

$$e \cdot K(f) = \mathtt{a} \cdot K(f) \leqq K(\mathtt{a} \cdot f) = K(e \cdot f)$$

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

Proof.

We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

- If $e = e_0 + e_1$, then derive as follows:

$$e \cdot K(f) = (e_0 + e_1) \cdot K(f)$$

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$*. Now* $e \leqq K(e)$*.*

Proof.

We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

- If $e = e_0 + e_1$, then derive as follows:

$$\begin{aligned} e \cdot K(f) &= (e_0 + e_1) \cdot K(f) \\ &\equiv e_0 \cdot K(f) + e_1 \cdot K(f) \end{aligned}$$

The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$. *Now* $e \leqq K(e)$.

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The round-trip theorem — right-to-left

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We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

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&\leqq K(e_0 \cdot f) + K(e_1 \cdot f) && \text{(IH)} \\
&\leqq K(e_0 \cdot f + e_1 \cdot f) \\
&\leqq K((e_0 + e_1) \cdot f)
\end{aligned}
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The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

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&\leqq K(e_0 \cdot f + e_1 \cdot f) \\
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The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

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We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

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The round-trip theorem — right-to-left

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The round-trip theorem — right-to-left

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The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

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We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

- If $e = e_0^*$, then we derive as follows:

$$K(f) + e_0 \cdot K(e \cdot f) \leqq K(f) + K(e_0 \cdot e \cdot f) \qquad \text{(IH)}$$

The round-trip theorem — right-to-left

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Let $e \in \mathbb{E}$*. Now* $e \leqq K(e)$*.*

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We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

- If $e = e_0^*$, then we derive as follows:

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The round-trip theorem — right-to-left

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We first claim that for all $f \in \mathbb{E}$, $e \cdot K(f) \leqq K(e \cdot f)$. Inductive cases:

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The round-trip theorem — right-to-left

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&\leqq K((1 + e_0 \cdot e) \cdot f) \\
&\leqq K(e \cdot f)
\end{aligned}
$$

  It then follows that $e \cdot K(f) = e_0^* \cdot K(f) \leqq K(e \cdot f)$.

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. *Now* $e \leqq K(e)$.

Proof.

We now have that $e \cdot K(f) \leqq K(e \cdot f)$ for all $f \in \mathbb{E}$.

The round-trip theorem — right-to-left

Lemma

Let $e \in \mathbb{E}$. Now $e \leqq K(e)$.

Proof.

We now have that $e \cdot K(f) \leqq K(e \cdot f)$ for all $f \in \mathbb{E}$.

We then conclude that

$$e \equiv e \cdot 1 \leqq e \cdot K(1) \leqq K(e \cdot 1) \leqq K(e)$$

□

The round-trip theorem

Theorem (Round-trip)

Let $e \in \mathbb{E}$. *Now* $e \equiv K(e)$.

Solutions to powerset automata

Lemma

Let $A = \langle Q, \rightarrow, I, F \rangle$ be an automaton and let $A' = \langle 2^Q, \rightarrow', \{I\}, F' \rangle$ be its powerset automaton. Furthermore, let s and s' be the least solutions to A and A' respectively. For $S \subseteq Q$ we have that $s'(S) \equiv \sum_{q \in S} s(q)$.

Solutions to powerset automata

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Proof sketch.

For $\leqq$: show that $t' : 2^Q \to \mathbb{E}$ given by $t'(S) = \sum_{q \in S} s(q)$ is a solution to A'.

Solutions to powerset automata

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Solutions to powerset automata

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Let $A = \langle Q, \rightarrow, I, F \rangle$ be an automaton and let $A' = \langle 2^Q, \rightarrow', \{I\}, F' \rangle$ be its powerset automaton. Furthermore, let s and s' be the least solutions to A and A' respectively. For $S \subseteq Q$ we have that $s'(S) \equiv \sum_{q \in S} s(q)$.

Proof sketch.

For $\leqq$: show that $t' : 2^Q \rightarrow \mathbb{E}$ given by $t'(S) = \sum_{q \in S} s(q)$ is a solution to A'.

For $\geqq$: show that if $s \in S \subseteq Q$, then s in A is simulated by S in A'.

From this, it follows that $s(q) \leqq s'(S)$, and hence $\sum_{q \in S} s(q) \leqq s'(S)$. □

Completeness

Theorem (Completeness)

Let $e, f \in \mathbb{E}$. *If* $[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}}$, *then* $e \equiv f$.

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Let $e, f \in \mathbb{E}$. *If* $[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}}$, *then* $e \equiv f$.

Proof.

Write s_A for the least solution to A (for $A \in \{A_e, A'_e, A_f, A'_f\}$). We derive as follows:

$$[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}} \implies L_{A_e}(e) = L_{A_f}(f)$$

Completeness

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Completeness

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Write s_A for the least solution to A (for $A \in \{A_e, A'_e, A_f, A'_f\}$). We derive as follows:

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[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}} &\implies L_{A_e}(e) = L_{A_f}(f) \\
&\implies L_{A'_e}(\{e\}) = L_{A_f}(\{f\}) \\
&\implies s_{A'_e}(\{e\}) \equiv s_{A'_f}(\{f\})
\end{aligned}
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Let $e, f \in \mathbb{E}$. *If* $[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}}$, *then* $e \equiv f$.

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&\implies L_{A'_e}(\{e\}) = L_{A_f}(\{f\}) \\
&\implies s_{A'_e}(\{e\}) \equiv s_{A'_f}(\{f\}) \\
&\implies s_{A_e}(e) \equiv s_{A_f}(f)
\end{aligned}
$$

Completeness

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Let $e, f \in \mathbb{E}$*. If* $[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}}$*, then* $e \equiv f$*.*

Proof.

Write s_A for the least solution to A (for $A \in \{A_e, A'_e, A_f, A'_f\}$). We derive as follows:

$$\begin{aligned}
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&\implies L_{A'_e}(\{e\}) = L_{A_f}(\{f\}) \\
&\implies s_{A'_e}(\{e\}) \equiv s_{A'_f}(\{f\}) \\
&\implies s_{A_e}(e) \equiv s_{A_f}(f) \\
&\implies K(e) \equiv K(f)
\end{aligned}$$

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Let $e, f \in \mathbb{E}$*. If* $[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}}$*, then* $e \equiv f$*.*

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[\![e]\!]_{\mathbb{E}} = [\![f]\!]_{\mathbb{E}} &\implies L_{A_e}(e) = L_{A_f}(f) \\
&\implies L_{A'_e}(\{e\}) = L_{A_f}(\{f\}) \\
&\implies s_{A'_e}(\{e\}) \equiv s_{A'_f}(\{f\}) \\
&\implies s_{A_e}(e) \equiv s_{A_f}(f) \\
&\implies K(e) \equiv K(f) \\
&\implies e \equiv f
\end{aligned}$$

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- Also gives us a decision procedure for $e \equiv f$!

Enjoy ESSLLI!